Looking ahead: expect to spend 4/11 + 4/18 on classical mechanics, then 3 lectures on stat mech, 4/25, 5/2, 5/9. (Note: 5/9 is exam period.) There will be one last HW (#6), distributed next week.

[Start 4/11 class with discussion "What choices of \( L(\mathbf{q}, \mathbf{\dot{q}}) \) should we permit?" — see Lecture 7 notes.]

Next topic: Hamiltonian viewpoint and its consequences.

Claim: In "any" Lagrangian \( L(\mathbf{q}, \mathbf{\dot{q}}) \), the associated evolution
\[
\frac{\partial}{\partial \dot{q}_i} \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial q_i} = 0
\]

can be written in a suitable coordinate system \((q_i, p_i)\) as
\[
\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}
\]

(Here \( L \) is not completely arbitrary — it should be convex in \( \dot{q} \) — growing faster than linear as \( |\dot{q}| \to \infty \).)
Recall from the Lecture 9 notes that a key example is:

\[ L = \frac{1}{2} \sum q_i (q_i \dot{q}_i - \dot{q}_i) \]

in which case:

\[ H = \frac{1}{2} \sum \left[ \dot{q}_i \right] p_i \cdot \dot{p}_i + U(q) \]

The general rule is \( p_i = \frac{\partial L}{\partial \dot{q}_i} \) and

\[ H(q, p) = \max \left\langle \dot{q}, p \right\rangle - L(q, \dot{q}) \]

i.e., \( H \) is the Legendre-Fenchel transform of \( L \) with \( \dot{q} \) holding \( q \) fixed.

Postponing the proof till later, let's discuss why this is interesting.

1st consequence: The "energy" \( H \) is constant along trajectories, since:

\[ \frac{dH}{dt} = \sum \frac{\partial H}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial H}{\partial \dot{q}_i} \dot{q}_i = 0 \]
continuing chain rule with Hamilton's

This is of course the same case law
we found in Lecture 9. There we
wrote it as

$$\left( \sum_i \frac{\partial}{\partial q_i} \right) \cdot L = \text{const along}
\text{trajectories.}$$

In light of $H$ (see (**) ), opt'y cancel is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

so

$$H(q, p) = \sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L(q, \dot{q})$$
as expected.

2nd consequence: "dimension reduction". If

$lagrangian$ is $indep$ of $\dot{q}_i$, then $p_i$ is constant +
problem reduces to solving Hamilton's eqns

in $\mathbb{R}^{2n-2}$ in $(q_1, \ldots, q_n, p_1, \ldots, p_n)$.

In fact $\frac{\partial p_i}{\partial \dot{q}_i} = -\frac{\partial H}{\partial q_i} = 0$ by hypoth. So $p_i = \text{const}$.

Thus: we can solve
\[
P_i = \frac{-\delta H}{\delta i}, \quad \dot{i} = \frac{\delta H}{\delta P_i} \quad (j = 2)
\]

By substituting certain values of \( p_i \) into \( \dot{H} \) (note \( \dot{H} \) is independent of \( g_{i1} \) by the proof). Finally, get \( \dot{g}_{i1} \) at the end by integrating

\[
\frac{d\dot{g}_{i1}}{dt} = \frac{\delta H}{\delta P_i}
\]

along the resulting path.

Proceeding any \( t \) can be repeated. So if \( \dot{g}_{i1} \) depends on just one spatial variable, evolution can be replaced to phase plane analysis.

3rd consequence: Liouville's Theorem. In the \((q, P)\) cords, the normal flow is volume-preserving on \( I \).

Ph. For any flow we can consider its "infinitesimal generator"

\[
\text{image of } x \text{ after the } t = \frac{x}{T} + \int_{0}^{t} \xi(\tau) d\tau + \theta(t^2)
\]

until generator
and flow is vol-pres if \( \text{div} f = 0 \) since
\[
\frac{\partial}{\partial t} \left( \text{vol of image of } D \right) = \int_D \frac{\partial}{\partial t} \left( \text{det } (I + tf) \right) \, dt_{|t=0} = \int_D \text{div } f
\]

Applying this to the Hamiltonian flow:
then \( \overline{x} = (\overline{\phi}, \overline{\rho}) \) and \( \overline{f} = \left( \frac{\partial H}{\partial \rho}, -\frac{\partial H}{\partial \phi} \right) \) so
\[
\text{div } f = \sum_i \frac{\partial}{\partial \phi_i} \left( \frac{\partial H}{\partial \phi_i} \right) - \frac{\partial}{\partial \rho_i} \left( \frac{\partial H}{\partial \rho_i} \right) = 0
\]

Lyapunov's Theorem places important constraints on the character of the flow, e.g. through Poincaré's recurrence theorem: if \( f \) is a vol-preserving map (e.g., the time-1 map of a Hamiltonian flow on phase space) \( + f (D) = D \) for some set of finite vol, then it is "recurrent" in seven that:

for any set \( B \) of positive measure (e.g. ball
\[ x \in B \Rightarrow f^n(x) \text{ is again in } B \] for some \( n < \infty \).
(Instructor example: root of $S'$ by rational or irrational angle.)

Proof of recurrence: clearly $B, g(B), g^2(B)$, etc cannot all be mutually disjoint, so $x \in g^k(B) \cap g^l(B)$, $l < k$. Then $x_0 = g^{-k}x$, satisfies $x_0 \in B \cap g^{-l}(B)$, so

\[ x_0 \in B + g^{k-l}x \in B \]

as asserted.

Surprising consequence:

- Consider motion of a ball in an asymmetrical bowl

Region of phase space $ST + U \times V$ is unit. So ball rolls to its initial position and velocity (almost).

[In fact, a bit worse yet above this must happen infinitely many times.]

- Consider motion of interacting particles modeling a gas in a container. Suppose initially all particles are in left half.
Then eventually, they'll revert to a similar state. (Seems impossible - contrary to 2nd law of thermodynamics. Resolt's idea: "eventually", in this case, is longer than lifetime of universe.)

Vol pres clear of this will be covered when we get to stat mechanics.

OK, now let's fulfill our promise: why does the lagr evolve eqn

$$\frac{\partial}{\partial t} \left( \frac{\partial H}{\partial \dot{q}_i} \right) - \frac{\partial H}{\partial q_i} = 0$$

become

$$\dot{p}_i = -\frac{\partial H}{\partial \dot{q}_i}, \quad \ddot{q}_i = -\frac{\partial H}{\partial q_i}$$

when we set

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and

$$H(\dot{q}, p) = \max \left< \dot{\phi}, p \right> - L(\dot{q}, \phi)$$
Some notes:

1. \( p_i = \frac{\partial L}{\partial \dot{q}_i} \) is a well-defined function of \( q \) and \( \dot{q} \).

To see the reason to work with \( (q,p) \) instead of \( (\dot{q},\dot{q}) \) is just a convenient change of variables. (Convexity of \( L \) over \( \dot{q} \Rightarrow \) then change of var is invertible)

2. The best \( \xi \) or data of \( H \) satisfies

\[
\dot{p}_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow \quad p_i = \frac{\partial L}{\partial \dot{q}_i}
\]

\[
p_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow H(q, p) = \langle \dot{q} \cdot p \rangle = L(q, \dot{q})
\]

Now, take differentials:

\[
\frac{dH}{dq_i} = \sum \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \quad \text{by Chain rule}
\]

and

\[
\frac{dH}{dq_i} = \sum \dot{q}_i dp_i + p_i \frac{\partial}{\partial q_i} - \sum \left( \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)
\]

by note (b)

So

\[
\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \Rightarrow \frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{as linear phase space,}
\]
(Warning: in calculating \( \frac{\partial H}{\partial \dot{q}_i} \), \( q_i \) is held fixed; in calculating \( \frac{\partial H}{\partial \ddot{q}_i} \), \( \dddot{q}_i \) is held fixed.)

Now finally, the Lagrange trajectory \( \frac{\partial H}{\partial \dot{q}_i} = \frac{2H}{\dot{q}_i} \)

has the property that along this curve

\[
\frac{d}{dt} \dot{q}_i = \ddot{q}_i = -\frac{\partial H}{\partial \dot{q}_i}
\]

and

\[
\frac{d}{dt} \dddot{q}_i = \dddot{q}_i = \frac{\partial H}{\partial \dddot{q}_i}
\]

as asserted.

Discussion about Legendre transform: it isn't only interesting in mechanics. It also has a foundation of convex duality, a useful tool in calculus of variations. Here's an example:

consider a convex variational problem

\[
\min_{u=u_0+2\delta} \int (u) \quad \text{subject to } \frac{\partial H}{\partial u} = \text{ convex}
\]

with \( \Phi : \mathbb{R} \to \mathbb{R} \) convex. The associated dual

\[
\max \int \Phi(u) \quad \text{subject to } \frac{\partial H}{\partial \dot{q}_i} = \text{ convex}
\]
problem is
\[
\max \left\{ (\sigma, u) : -\frac{1}{2} \int \phi^*(\sigma) \right\}
\]
dw, \sigma = 0 \in \Omega

where \( \phi^*(\sigma) = \max_{\xi} \left\{<\sigma, \xi> - \phi(\xi)\right\}. \)

The plans are related by
\[
(\forall) \int_{\Omega} (\sigma, u) - \frac{1}{2} \int \phi^*(\sigma) \leq \frac{1}{2} \int \phi(\sigma)
\]

whenever dw, \sigma = 0 \in \Omega + 2u = u_0 at \partial \Omega
(i.e. for any \"aliveness\" \(\sigma, u\))

(**) values are equal when \(\sigma + u\) are optimal for their respective problems

\(\text{Pf} (\forall)\): elementary, using def of \(\phi^*\) and integrate by parts

\(\text{Pf} (**)\): also elementary, at least if \(\phi\) is smooth since then \(\sigma = \frac{\partial \phi}{\partial \sigma}\)
\(u = \text{optimal choice}\)
in order of the \"\(\sigma\) problem\".