Mechanics - Lecture 8, 3/21/2012

Note: no class Wed 4/4 (and no office hrs either).

Fresh start today: first of several lectures on "classical mechanics." Today = quick sweep of basics for interacting particles in $\mathbb{R}^n$. Then a couple of lectures about Lagrangian + Hamiltonian viewpoints (with links to Calc of Variations, geometry, and more)

Syllabus includes list of recommended books (on reserve). I esp like

- B"uchler's short lect notes volume - too terse to really learn from, but good for a 2nd pass

- Arnold's book - still a bit terse, and mixes basic topics with esoteric ones, but nevertheless elegant (also, few exercises + terse examples)

- Jose + Saletan - truly written as a textboo (for physics students) so a good source of examples + problems
Focus initially on **basic examples** involving one or more bodies in $\mathbb{R}^3$ (or $\mathbb{R}^2$ or $\mathbb{R}^1$):

\[ m_i \ddot{x}_i = \mathbf{f}_i \]

\( m_i \) = mass of \( i \)th body

\( \mathbf{f}_i \) = force on \( i \)th body

\( \ddot{x}_i \) = acceleration of \( i \)th body

We'll usually focus on **conservative forces**:

\[ \mathbf{f}_i = -\frac{\partial U}{\partial \mathbf{x}_i} \quad \text{where} \ U = U(x_1, \ldots, x_N) \]

\( U \) is the "potential energy"

Note (from calculus) that \( \mathbf{f} = (f_1, \ldots, f_N) \) is conservative if associated "work"

\[ \int_{\text{initial}}^{\text{final}} \sum_i f_i \, dx_i \]

is path-independent (it then equals the change in \( U \)). Key feature of conservative forces is equal to the "total energy"

\[ H = \frac{1}{2} \sum_i m_i \dot{x}_i^2 + U \]

\( H \) = total energy

\( U \) = potential energy
is conserved, since

\[ \frac{dE}{dt} = \sum m_i \dot{x}_i \dot{\dot{x}}_i + \left( \frac{\partial V}{\partial x_i} \right) \ddot{x}_i = 0 \]

when (**) and (***) hold.

Example 1: a single particle in a central force field. Eq. in 3D is

\[ m \ddot{x} = -\frac{\partial V}{\partial r} \]

but it reduces to a single ODE for \( r = 1 \times 1 \), namely

\[ m \ddot{r} = -\frac{\partial \Phi}{\partial r} \]

Gravitation is the special case \( \Phi = -\frac{k}{r} \), giving

\[ m \ddot{r} = -\frac{k}{r^2} \]

Example 2: many particles interacting by pairwise attraction/repulsion

\[ m \ddot{x}_i = -\sum_{j \neq i} \frac{\partial V}{\partial x_i} \nabla (x_i - x_j) \]

Widely used example: the Lennard-Jones
Potential
\[ V(r) = c \left[ \left( \frac{\sigma}{r} \right)^2 - \left( \frac{\sigma}{r} \right)^6 \right] \]

where \( \sigma, c \) are constants. Graph looks like

![Graph](image)

where \( r_e \) solves \( \left( \frac{\sigma}{r_e} \right)^6 = 1/2 \). Particles repel when \( r < r_e \), attract when \( r > r_e \). Repulsion is strong as \( r \to 0 \) but attraction is weak as \( r \to \infty \).

Example 2.5 (similar to Ex 2 but not a special case): 3 particles interacting by gravity. This \( \Rightarrow \)

\[ M_i \ddot{X}_i = -2U \]

where \( U = -\frac{m_i m_j}{|X_i - X_j|} - \frac{m_i m_k}{|X_i - X_k|} - \frac{m_j m_k}{|X_j - X_k|} \)

(Different from Ex 2 since potential is proportional to \( m_i m_j \))
Example 3: particle attached to 1D linear spring

\[ m \ddot{x} = -2x \quad x \in \mathbb{R} \]

Exactly solvable of course: \( x(t) = A \sin(\omega t + \phi) \),
\( \omega^2 = 27/m \).

Explicit solutions are useful:

a) Because they help our gain intuition

b) Because we can perturb around them (e.g., solar system is a perturbation of 2-body plan involving just Earth + Sun).

But explicit solutions are rare! Aside from linear cases (like Ex 3), most accessible examples of exact solutions are:

1. Two body plan (e.g., Kepler's laws, when interaction is gravitational) - we'll discuss this later.

2. Systems with one spatial degree of freedom (e.g., "planetary pendulum")
\[ \ddot{x} = -\sin x \]

Let's discuss (2) row: key test is phase plane analysis. Evolution eqn defines a flow in 2D "phase plane" \((x, \dot{x})\). If force is conservative (always true in 1D provided \(f\) depends only on \(x\), not \(\dot{x}\)) then

\[ H = \text{kinetic + potential energy} \]

is conserved, so system moves on 1D curve \(H = \text{const}\) in 2D space of \((x, \dot{x})\). It is thus relatively easy to visualize what happens (even if explicit form isn't available).

Example: \( \ddot{x} = -U'(x) \), where \( E = \int U(x) \) with

\[ U(x) = \begin{cases} \infty & \text{if } x < a \text{ or } x > b \\ 0 & \text{otherwise} \end{cases} \]

\[ H = \frac{1}{2} \dot{x}^2 + U(x) = E \]

\( E > E_2 \)

Local max of \( U \) is unstable and \( \text{pt} \)
local min of \( U \) are stable (but not asymptotically stable).
Consider a closed system of interacting particles.

The system is closed if the only forces present are those associated to pairwise interactions (via pair-wise potentials $U = \sum U_{ij}(x_i - x_j)$). In such a system,

$$m_i \ddot{x}_i = \sum_{j \neq i} F_{ij} \quad \text{where} \quad F_{ij} = \frac{d}{dx_j} U_{ij}$$

Key property of such systems: linear momentum is conserved. Here

$$\text{linear momentum} = \sum_{i=1}^{N} m_i \dot{x}_i \quad \text{(a vector)}$$

and proof of cons. is elementary:

$$\frac{\partial}{\partial t} \sum_i m_i \dot{x}_i = \sum_i m_i \ddot{x}_i = \sum_i \sum_{j \neq i} F_{ij} = 0$$

since $F_{ij} = -F_{ji}$.

Implicit corollary: center of mass has accel 0 in such a system. (Since law of motion is invariant w/ respect to redefining a linear system,
Another key property of closed systems: angular momentum is also conserved. Here:

\[ \text{angular momentum} = \sum_i x_i \times m_i \dot{x}_i \]

(vector cross-product in \( \mathbb{R}^3 \)). It's again elementary:

\[ \frac{d}{dt} \sum_i x_i \times m_i \dot{x}_i = \sum_i x_i \times m_i \ddot{x}_i \]

\[ = \sum_i \sum_{j \neq i} x_i \times F_{ij} \]

For any pair \( i \neq j \),

\[ F_{ij} = \lambda(x_j - x_i) \quad \Rightarrow \quad F_{ij} = -\lambda(x_i - x_j) \]

\[ x_i \times F_{ij} + x_j \times F_{ji} = \lambda x_i \times (x_j - x_i) - \lambda x_j \times (x_i - x_j) \]

\[ = \lambda x_i \times x_j + \lambda x_j \times x_i = 0. \]

Thus, the entire sum = 0.

Corollary: If a single particle in \( \mathbb{R}^3 \) moves in a central force field, then it remains in the plane determined by its initial position.
Repeat the proof of case of angular momentum (this time \( \mathbf{x} \parallel \mathbf{x}_1 \)) or else recognize that this is like the case of 2 particles with \( m_2 = \infty \). Since \( H = \mathbf{x}_1 \cdot \mathbf{p}_1 \) is constant, \( x_1(t) \) and \( x_2(t) \) remain in the plane \( \mathbf{1} \cdot \mathbf{H} \).

I proposed to discuss the 2-body plan. Let's do that now:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= -\frac{\partial U}{\partial x_1}, \\
    m_2 \ddot{x}_2 &= -\frac{\partial U}{\partial x_2}
\end{align*}
\]

where \( U = U(1x_1 - x_2) \).

Claim: \( \mathbf{z} = x_1 - x_2 \) satisfies

\[
\frac{m_1 m_2}{m_1 + m_2} \ddot{z} = -\frac{1}{2} \sqrt{U(1z_1)}
\]

ie it evolves like motion of a single particle of mass \( \frac{m_1 m_2}{m_1 + m_2} \) in the central force field close to \( U \). (Set \( \overline{m} = \frac{m_1 m_2}{m_1 + m_2} \))
2. z remains in a plane.

3. in polar coords \( H = \Phi(t), r^2 \) is constant (this is the out-of-plane component of angular momentum) and \( r(t) \) behaves like a 1D particle with potential energy

\[
V(r) = U(r) + \frac{\mu^2}{2r^2}
\]

Since we understand 1D systems well (using phase plane analysis) clearly (3) lets us analyze the system more or less completely. [For special case of gravitational interaction, a little extra work \( \Rightarrow \) Kepler's laws, cf. Arnold Chap 2.4 or Jose &Saletan Chap 2.3]

Explain the claims:

About (2) we expect reduction to a on-particle system, since motion of center of mass is trivial. Actually, if it is elementary it does not use cons of momentum: mult X, -epn by \( m_2 \), \( X, -epn by m_1 \), subtract to get...
\[ m_2 \dot{m}_1 \ddot{x}_1 = -m_2 \ddot{z} U \quad \dot{m}_1 \dot{m}_2 \ddot{x}_2 = m_1 \ddot{z} U \]
\[ \Rightarrow \dot{m}_1 \dot{m}_2 \ddot{z} = -(m_1 + m_2) \ddot{z} U (121) \]
as asserted.

About (2): Cons of momentum (combined with (1)) shows that \( z \) stays in a plane.

About (3): if \( r, \phi \) are polar co-ords in plane of motion, let

\[ \hat{\xi} = (\cos \phi, \sin \phi) = e_r \]
be radial unit vector

and

\[ e_\phi = (-\sin \phi, \cos \phi) = e_\perp \]
be orthogonal to it.

By calculus,

\[ \dot{z} = \dot{r} e_r + r \dot{\phi} e_\phi \]

Cons of momentum says \( \dot{z} \wedge \dot{z} \) is constant.

But this is \( r e_r \wedge (r \dot{e}_r + r \dot{\phi} e_\phi) = r^2 \dot{\phi} e_r \times e_\phi \).

So \( r^2 \dot{\phi} \) is constant.

Differentiating further, using \( \dot{e}_r = \dot{\phi} e_\phi \)

\[ \dot{e}_\phi = -\dot{\phi} e_r \]
we get

\[ \ddot{z} = \dot{r} e_r + \dot{\phi} \dot{e}_\phi + \ddot{r} e_r + \dddot{\phi} e_\phi - r \dddot{\phi} \dot{e}_\phi \]

\[ \Rightarrow \ddot{z} = (\dddot{r} - r \dddot{\phi}^2) e_r + (\dddot{\phi} + r \dddot{\phi}) e_\phi \]

Since the force field is central,

\[ \bar{m} (\dddot{r} - r \dddot{\phi}^2) = -U'(r), \quad \dddot{\phi} + r \dddot{\phi} = 0 \]

But \( \dddot{\phi} = \frac{\bar{H}}{r^2} \) (recall: \( H \) is constant determined by initial conditions). So

\[ \bar{m} \dddot{r} = -U'(r) + \bar{m} r \frac{\bar{H}^2}{r^4} \]

\[ = -U'(r) - V'(r) \quad \text{with} \quad V = \frac{\bar{m} \bar{H}^2}{2 r^2} \]

You probably knew that for the gravitational law \( U(r) = -\frac{k}{r} \) the orbits are all ellipses (if bound) or hyperbolas (if unbounded). But that is very special to the inverse-square gravitational law. How can it be used to get qualitative information on a robust
way? Use phase plane analysis! Choose several constants $= 1$ for simplicity, we're left to consider

$$U_{eff}(r) = \frac{1}{2} r^2 - \frac{1}{r}$$

grow potential extra term with $k = 1$

$V(r)$ with $\bar{m} = M = 1$

Trajectories in phase plane are arose

$$\frac{1}{2} \dot{r}^2 + U_{eff}(r) = E \quad \text{constant}$$

These curves look like

closed orbits for $E < 0$; unbounded orbits for $E > 0$; there's one orbit at which $r$ is constant and $\dot{r} = 0$ - it corresponds to $E = \text{min value of } U_{eff}$. 