## MECHANICS - Problem Set 4, assigned 3/16/2022, due 4/5/2022

These problems are concerned with linear elasticity. Problem 1 asks you to explain why an isotropic Hooke's law is described by just two constants. Problems 2 and 3 explore Korn's inequality. Problem 4 gives an example of an elasticity problem with an explicit separation-of-variables solution. Problems 5 explores why linear plate theory involves solving a fourthorder PDE. Elasticity problems are often solved numerically by minimizing the variational principle in a finite-dimensional class of functions; problem 6 considers the accuracy of the result.

1. Elastic symmetries. A linearly elastic material is symmetric under a rotation $R$ if its Hookes' law satisfies $\alpha\left(R^{T} e R\right)=R^{T} \alpha(e) R$. Show, by a direct argument, that if this holds for any $R \in S O(3)$ then $\alpha e=2 \mu e+\lambda(\operatorname{tr} e) I$ for some constants $\lambda, \mu$. (Hint: start by showing that $\sigma=\alpha e$ must be simultaneously diagonal with e.)
2. Korn's inequality for periodic deformations. Korn's inequality for periodic deformations says

$$
\int_{Q}|\nabla u|^{2} d x \leq C \int_{Q}|e(u)|^{2} d x
$$

when $u: R^{n} \rightarrow R^{n}$ is periodic in each variable with period 1 and $Q=[0,1]^{n}$ is the unit cell. Give a proof using the Fourier representation of $u$. What is the best possible value of the constant $C$ ? Why is there no condition about $\int \nabla u$ being symmetric?
3. Korn's inequality for beams. Let $\Omega_{h} \subset R^{2}$ be the long, thin domain $\{0<x<$ $1,-h / 2<y<h / 2\}$ where $h \ll 1$. Korn's second inequality for this domain says

$$
\int_{\Omega_{h}}|\nabla u|^{2} d x \leq C(h) \int_{\Omega_{h}}|e(u)|^{2} d x \quad \text { provided } \int_{\Omega_{h}} \nabla u \text { is symmetric. }
$$

(a) Show that $C(h)$ must be at least of order $h^{-2}$, by considering deformations of the form $u(x, y)=\left(-y \phi_{x}, \phi\right)$ where $\phi=\phi(x)$.
(b) Show that the inequality is true with $C_{h} \sim h^{-2}$. You may assume (for simplicity, this is not really necessary) that $1 / h$ is an integer. Hint: divide $\Omega_{h}$ into $1 / h$ squares of side $h$. Korn's inequality (for squares) controls $\nabla u-\left(\begin{array}{cc}0 & \omega_{j} \\ -\omega_{j} & 0\end{array}\right)$ on the jth square in terms of the strain on that square, for some $\omega_{j} \in R$. Use Korn's inequality again (this time for rectangles of eccentricity 2) to control $\omega_{j}-\omega_{j-1}$ in terms of the strain on the ( $\mathrm{j}-1$ )st and j th squares. Then apply a discrete version of Poincare's inequality in one space dimension to control the variation of $\omega_{j}$ with $j$.
(c) How do you think these results would extend to a thin plate-like domain $\{0<$ $x<1,0<y<1,-h / 2<z<h / 2\}$ in $R^{3}$ ? (Just discuss how the 3D problem is similar or different; I'm not asking for a complete solution.)
4. Separation of variables. Let $\Omega$ be a "ball with a hole removed":

$$
\Omega=\left\{x: \rho^{2}<|x|^{2}<1\right\} .
$$

Suppose it is filled with an isotropic, homogeneous, linearly elastic material with Lamé moduli $\lambda$ and $\mu$, and constant pressure $P$ is applied at the outer boundary $|x|=1$. The inner boundary $|x|=\rho$ is traction-free. Find the displacement $u(x)$ and the associated stress $\sigma(x)$ using separation of variables.
5. Bending of a thin plate. Consider now a thin, constant-thickness plate whose midplane occupies a region $D$ in the $x-y$ plane. The upper and lower surfaces are $z=$ $\pm h / 2$, so the thickness is $h$. Consider a deformation of the form $u=\left(-z \phi_{x},-z \phi_{y}, \phi+\right.$ $\left.\frac{\alpha}{2} z^{2} \Delta \phi\right)$. Find the associated strain and stress, keeping only terms of order $h$. Show that for the faces to be traction-free (to this order) we need $\alpha=\lambda /(\lambda+2 \mu)$. Do the $z$ - integrations in the basic variational principle, to obtain the elastic energy as an integral over $D$. Notice that it involves second derivatives of $\phi$, so the associated PDE is a fourth-order equation!
6. Minimization over a finite dimensional subspace. Elasticity problems are often solved numerically using the Finite Element Method, which amounts to minimizing the associated variational principle over a finite-dimensional class of functions (for example, those that are continuous and piecewise linear on some triangulation of the domain). Let's explore how well this works by considering the case of an elastic body with Hooke's law $A$, loaded by force per unit volume $f$ and clamped at its boundary. If the body occupies the region $\Omega \subset \mathbb{R}^{3}$ then we know that the solution $u_{*}$ minimizes

$$
E[u]=\int_{\Omega} \frac{1}{2}\langle A e(u), e(u)\rangle-u \cdot f d x
$$

subject to the constraint $\left.u\right|_{\partial \Omega}=0$. Let's call the minimizer (the solution of this elasticity problem) $u_{*}$.

Now suppose $V$ is a finite-dimensional subspace of the admissible functions (that is: each $v \in V$ satisfies $\left.\left.v\right|_{\partial \Omega}=0\right)$.
(a) Show that for any $v \in V$,

$$
\begin{align*}
E[v]-E\left[u_{*}\right]= & \int_{\Omega}\left\langle A e\left(u_{*}\right), e\left(v-u_{*}\right)\right\rangle-\left(v-u_{*}\right) \cdot f d x \\
& +\int_{\Omega} \frac{1}{2}\left\langle A e\left(v-u_{*}\right), e\left(v-u_{*}\right)\right\rangle d x \tag{1}
\end{align*}
$$

(Hint: this is just a matter of algebraic manipulation, using the quadratic character of the elastic energy.)
(b) Show, using the PDE satisfied by $u_{*}$, that the first term on the right of (1) vanishes.
(c) Conclude, using Korn's inequality and the positive definiteness of the Hooke's law as a quadratic form, that

$$
\begin{equation*}
\int_{\Omega} \int_{\Omega}\left|\nabla\left(v-u_{*}\right)\right|^{2} d x \leq C\left[E(v)-E\left(u_{*}\right)\right] . \tag{2}
\end{equation*}
$$

(Explaining the importance of this estimate: to find a good approximate solution in $V$, we must clearly begin by assuming that $V$ is rich enough to contain a function near $u_{*}$. Making this precise: we should assume that $V$ contains at least one $v$ for which $E[v]$ is close to $E\left[u_{*}\right]$. Then (2) says that minimizing the variational principle in $V$ - which minimizes the right hand side of (2) - is guaranteed to achieve a good approximation of $u_{*}$ in the $H^{1}$ norm.)

