MECHANICS – Problem Set 2, distributed 2/8/22, due 2/22/2022

(1) Consider a one-dimensional inextensible rod in the plane with no body load, and assume the constitutive law $M(s) = A\theta_s$. If the rod has length L, then our reference interval can be $(0, L)$.

(a) Suppose $\theta(0) = \theta_0$ and $\theta(L) = -\theta_0$ for some $\theta_0 \in (0, \pi/2)$, and assume no forces are applied at the ends of the rod. Show that the rod forms a piece of a circle, and that its two ends lie at the same height. What is the radius of the circle?

(b) Now suppose the rod sits on a table (so the two ends lie at the same height), but rather than specifying θ at the ends the rod has equal and opposite horizontal forces holding it in place (force $(-\lambda, 0)$ at the right end and force $(\lambda, 0)$ at the left end), with no applied bending moment ($\theta'(0) = \theta'(L) = 0$). What differential equation and boundary condition does $\theta(s)$ solve? Show (using the differential equation, but without actually solving it) that the ends do indeed lie at the same height. Does this rod form a piece of a circle? (Why or why not?)

(2) **The bending stiffness of xerox paper.** Recall our discussion of "the xerox paper problem" from Lecture 2: consider a standard 8.5×11 sheet of paper, held at one edge so the tangent there is vertical. We showed that if $r(s) = (\cos\theta(s), \sin\theta(s), 0)$ describes its profile then

$$A\theta'' + f_0 s \cos\theta(s) = 0$$

on $0 < s < L$, with boundary conditions

$$\theta'(0) = 0, \quad \theta(L) = -\pi/2,$$

where $s = 0$ corresponds to the free edge and $s = L$ corresponds to the edge being held. Here L has dimensions of length (for standard xerox paper it is 11 inches) and A/f_0 has dimensions of $(\text{length})^3$ (this is clear from the equation, since θ is dimensionless and s has dimensions of length). Evidently, $\alpha = \frac{A}{f_0 L^3}$ is dimensionless. *Estimate the value of α for a standard sheet of xerox paper.* [*Comment*: I expect a ballpark estimate, not an exact answer. I know at least two methods: one is to nondimensionalize the differential equation and solve it numerically for different choices of $\alpha = \frac{A}{f_0 L^3}$, then compare the result to what a sheet of paper does; the other uses Problem 4.17 of Howell-Kozyreff-Ockendon. You are, of course, only expected to offer one solution.]

(3) **A variational perspective on bifurcation of the elastica.** Recall from the Lecture 2-3 notes that equilibrium configurations of the elastica (with length 1 and the physical constant A set to 1) are critical points of the functional

$$E[\theta] = \int_0^1 \frac{1}{2}\theta_s^2 + \lambda\cos\theta \, ds,$$

and that (to leading order) the bifurcation diagram is described by $\theta(s) = g\phi(s)$ with

$$\lambda - \lambda_1 = \frac{\pi^2}{32} g^2 \tag{1}$$

where $\phi(s) = \sin\left(\frac{\pi}{2}s\right)$ and $\lambda_1 = \pi^2/4$. Give another "derivation" of (1) by (i) assuming that $\theta(s) = g\phi(s)$ for some g, (ii) estimating $E[\theta]$ as a function of g, using the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$, then (iii) considering the condition that g be a critical point of the resulting expression. (I put "derivation" in quotes, because a proper explanation why it's sufficient to consider $\theta = g\phi$ requires the analysis that's behind Liapunov-Schmidt reduction.)

(4) **Bifurcation of an imperfect elastica.** Consider an imperfect elastica, with (constant) intrinsic curvature δ. This means the constitutive law is $m_3 = A(\theta' - \delta)$. We take the length to be 1, and the boundary conditions to be the same as considered in Lecture 2: the left side ($s = 0$) is clamped in a horizontal position, while the right side ($s = 1$) is loaded horizontally. For simplicity we set $A = 1$.

(a) Show that the associated boundary value problem is

$$\theta'' + \lambda \sin\theta = 0, \quad \theta(0) = 0,\ \theta'(1) = \delta.$$

(b) Show that solutions of this boundary-value problem are critical points of

$$E = \int_0^1 \tfrac{1}{2}(\theta' - \delta)^2 + \lambda\cos\theta\, ds$$

subject to boundary condition $\theta(0) = 0$. (Note that I have not imposed $\theta'(1) = \delta$; you must explain why a critical point satisfies this "natural boundary condition.")

(c) Consider the associated linear problem

$$u'' + \lambda_0 u = f, \quad u(0) = 0,\ u'(1) = g$$

with $\lambda_0 = \pi^2/4$. Show that for a solution to exist, the data must satisfy $\int_0^1 f(s)\phi(s)\, ds = g$ with $\phi(s) = \sin(\frac{\pi}{2}s)$. [More is true: when this condition holds a solution exists, and is unique up to an additive multiple of $\phi(s)$. You'll need this in part (d); I'm not asking you to prove it, but if you've taken PDE then you should know how to give a proof.]

(d) Seek a formal solution for the configuration of the buckled structure by means of a perturbation expansion

$$\begin{aligned}
\theta &= 0 + \epsilon\theta^{(1)} + \epsilon^2\theta^{(2)} + \dots \\
\delta &= 0 + \epsilon\delta^{(1)} + \epsilon^2\delta^{(2)} + \dots \\
\lambda &= \pi^2/4 + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \dots
\end{aligned}$$

Reconcile your answer with your physical intuition about which way the elastica should buckle (depending on the sign of δ).

(e) Liapunov-Schmidt reduction says that the equilibrium equation can be expressed in the form

$$f(x, \mu; \delta) = 0$$

with the notation

$$\begin{aligned}\theta &= x\phi + \tilde{\theta}, \quad \tilde{\theta}\perp\phi \\ \mu &= \lambda - \pi^2/4.\end{aligned}$$

Show that your answer to (d) is consistent with f having a Taylor expansion near 0 of the form

$$f(x,\mu;\delta) \approx x^3 + c_1\mu x + c_2\delta$$

for suitable choices of the constants c_1 and c_2.

(f) Give a variational perspective on this problem, analogous to the one requested in Problem 2 for the case $\delta = 0$.