MECHANICS – Problem Set 1, distributed 1/25/2022, due 2/8/2022

These problems involve elastic strings. Their goal is to develop intuition about "stress" and "strain" in this most elementary setting. (You might find it helpful to look at Antman's book, which discusses these topics and much more.)

Basic notation for extensible strings: the reference configuration is $[0, L]$. The deformation is represented by $\mathbf{r} : [0, L] \to R^3$. The strain is $\nu = |\mathbf{r}_s|$. The stress is $\mathbf{n} = N(\nu, s)\mathbf{r}_s/|\mathbf{r}_s|$. The equation of equilibrium is $\mathbf{n}_s + \mathbf{f} = 0$. We call $N(\nu, s)$ the constitutive law of the string, and $\mathbf{f}$ the distributed load.

(1) The catenary problem: constant vertical load per unit reference length. In this problem we take $\mathbf{f} = f_0(0, 0, -1)$, representing the effect of gravity on a string of uniform density.

(a) Consider a string with the constitutive law $N(\nu, s) = \nu$. [This doesn't satisfy the restriction $N(1, s) = 0$, and it is positive for $0 < \nu < 1$; therefore we cannot use it for strings in compression, nor when the strain is close to 1. But it is nevertheless plausible that a real string might satisfy such a law, at least approximately, when the tension is large.] Suppose $\mathbf{r}(0) = (0, 0, 0)$ and $\mathbf{r}(L) = (a, 0, 0)$ with $|a| \gg L$. Show that the hanging string forms a parabola.

(b) This is the analogue of part (a) for compression. Consider the constitutive law $N(\nu, s) = -1/\nu$. Suppose $\mathbf{r}(0) = (0, 0, 0)$ and $\mathbf{r}(L) = (a, 0, 0)$ with $|a| \ll L$. Show there is a compressive equilibrium in which the string forms an arch, and give an equation for the shape of the arch. [There is more than one way to approach this problem. One is to look for a solution such that $r_s/|r_s| = (\cos\theta(s), 0, \sin\theta(s))$, with $r(0) = (0, 0, 0)$, $\theta(0) = \theta_0$ at $s = 0$ and $r(L) = (a, 0, 0)$, $\theta(L) = -\theta_0$ at $s = L$, where θ_0 is a constant between 0 and $\pi/2$. It's relatively easy to see how $\theta(s)$ should depend on s, but you must work a bit to find how θ_0 depends on the parameters that were originally specified (L, f_0, and a), and to show that it meets the consistency condition $|r_s| \ll 1$ under suitable conditions on L, f_0, and a.]

(c) Now consider an *inextensible* string. This means we impose the constraint $|\mathbf{r}_s| = 1$. The stress is then $\mathbf{n}(s) = N(s)\mathbf{r}_s$. We no longer have the right (or the need) to specify a constitutive law for N: it is determined by the equation of equilibrium. Suppose $\mathbf{r}(0) = (0, 0, 0)$ and $\mathbf{r}(L) = (a, 0, 0)$ with $|a| \leq L$. Show there is a unique tensile equilibrium (hanging down), and a unique compressive configuration (arching up). Determine their shapes. [Hint: start by observing that a tensile solution $r(s, N(s)$ determines a compressive one $\tilde{r}(s) = (r_1(s), r_2(s), -r_3(s)), \tilde{N}(s) = -N(s)$ and vice versa, so it suffices to consider the tensile case. As in part (b), more than one approach is possible. One proceeds along the lines suggested for (b), looking for $r_s = (\cos\theta(s), 0, \sin\theta(s))$ with $\theta(0) = \theta_0$ and $\theta(L) = -\theta_0$ for some θ_0 between $-\pi/2$ and 0; you must then get an equation for $\theta(s)$ and look at how θ_0 is related to the original parameters of the problem. An entirely different approach is to consider the curve $(r_1(s), 0, r_3(s))$ as the graph of a function $(x, 0, y(x))$, and to show that the function y (which is defined on the interval $(0, a)$, with $y(0) = y(a) = 0$) satisfies $d^2y/dx^2 = \gamma\sqrt{1 + (dy/dx)^2}$ for some constant γ; the solution has a closed-form expression involving the hyperbolic cosine (this curve is known as a catenary).]

Comment. The situation suggested by these examples is in fact more general. An extensible string loaded by gravity has a unique tensile equilibrium (which hangs down). There can be multiple compressive equilibrium (which arch up). See Section III.3 of Antman. The compressive equilibria are of course unstable – you'll never get a string to form an arch – but they can be stabilized by even a little bending resistance, so they are preferred shapes for elastic arches.

(2) The suspension bridge problem: specified vertical load per unit horizontal length. In this problem we take $\mathbf{r}(s) = (x(s), 0, z(s))$ and $\mathbf{f} = (0, 0, -g(x)dx/ds)$.

(a) Explain why "specified vertical load per unit horizontal length" is correctly modelled by the proposed form for $\mathbf{f}$.

(b) Show that no matter what constitutive law the string has, every solution of the equilibrium equation must trace a curve of the form $\lambda dy/dx = \mu + G(x)$ where λ, μ are constants and $G' = g$. Thus, for example, if g is constant (typical of a real suspension bridge) the string lies along a parabola.

(c) Examine the special case of a point mass hanging from a string, ignoring the effects of gravity, by letting G approach a Heaviside function (so g approaches a "delta-function").

Comment. Though part (b) determines the "shape" of the string, it does not fully determine the number of solutions or the form of the map $s \to \mathbf{r}(s)$. Counting solutions amounts to asking how many pairs λ, μ can arise; finding $\mathbf{r}(s)$ requires use of the constitutive law. For a uniform string hanging from points of equal height, there is always exactly one tensile solution. There can be multiple compressive solutions. See Section III.4 of Antman.

(3) A uniform ring under constant pressure. Suppose the string represents the cross-section of a hollow fluid-filled cylinder. (We ignore the mechanical connections between different cross-sections.) Then s ranges over a circle rather than an interval, and it is natural to restrict the deformation to be planar: $\mathbf{r}(s) = (x(s), y(s))$. Let $e(s) = \mathbf{r}_s/|\mathbf{r}_s|$ be the unit tangent vector at $\mathbf{r}(s)$, and let $e^\perp = (e_2, -e_1)$ be the unit normal. We may suppose the string is parameterized so that e points counterclockwise and $e^\perp$ points outward. Uniform pressure p is represented by taking $\mathbf{f} = p\nu e^\perp(s)$, where p is a constant; assume the force has this form. Assume in addition that the string is uniform, i.e. the constitutive law is independent of s.

(a) Explain the presence of the factor ν in the formula for f. (Hint: "uniform pressure" really means uniform pressure per unit what?)

(b) Suppose the constitutive law satisfies $N'(\nu) > 0$ (so N is a monotonically increasing function of ν). Show that an equilibrium configuration must be circular, with constant strain. Moreover it is in tension if $p > 0$ and in compression if $p < 0$.

(c) Show that when p is large there may be multiple solutions, or none at all. (Do this by giving examples of reasonable constitutive laws with each type of behavior).

(d) Returning to part (b): show (by constructing a suitable example) that if the map $\nu \mapsto N(\nu)$ is not monotone, then there can (for some p) be a solution with nonconstant strain.

Comment. This example differs from the catenary problem in that the load depends significantly on the deformed position of the string: a larger cylinder has more surface area and therefore experiences a greater total pressure. The possible nonexistence of tensile equilibria is a reflection of this fact. Section III.5 of Antman proves that there is always a unique compressive equilibrium, and discusses how the existence and multiplicity of tensile solutions is related to the behavior of the constitutive law.