Mechanics - Lecture 7 - 3/8/2022

These notes provide a variational perspective on linear elasticity. From a big-picture perspective: most things are similar to analogous results for Laplace's eqn ($-\Delta u + f = 0$ with suitable bc), but we need Korn's inequality in addition to Poincaré's inequality.

Recall from Lecture 6: if Hooke's law is A (a 4th order tensor, determining a quadratic form on symmetric matrices) then equil law is

$$\mathrm{div}(A\,e(u)) + f = 0 \quad ;$$

elastic energy is $\int_\Omega \frac{1}{2}\langle A e(u), e(u)\rangle$;

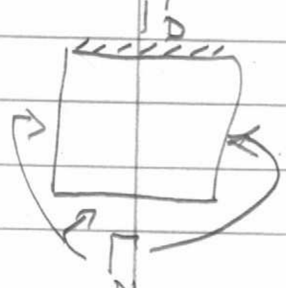
and a traction bc has the form

$$\sigma \cdot n = g \quad \text{at } \partial\Omega, \text{ where } \sigma = A e(u).$$

Uniqueness. In nonlinear elasticity, solns of the equilibrium eqns can easily be nonunique. (For example, a slender body loaded in compression can buckle.) Linear elasticity is different: the elastic energy is quadratic + convex (assuming $\langle A\xi, \xi \rangle \geq C|\xi|^2$ for any $\xi \neq 0$) + work done by load is linear, so any equil is at least a local min.

Strict convexity would make uniqueness obvious; but elastic energy is not strictly convex.

But recall this simple pf of uniqueness for Laplace's eqn with a Dirichlet bc: The soln of



$$\Delta u + f = 0 \quad \text{in } \Omega$$
$$u = u_0 \quad \text{at } \Gamma_D \subset \partial\Omega$$
$$\partial u / \partial n = g \quad \text{at } \Gamma_N \subset \partial\Omega$$

(where $\partial\Omega$ is decomposed into two disjoint parts, Γ_D and Γ_N).

is unique provided Γ_D is a set of positive surface measure. Proof: Subtract 2 solns $\Rightarrow$ must show that if u_0, f, and g are 0 then $u \equiv 0$.

Multiply eqn by u + integrate by parts:

$$\int_{\partial\Omega} u \underbrace{\frac{\partial u}{\partial n}}_{0} - \int_{\Omega} |\nabla u|^2 = 0$$

But $\nabla u \equiv 0 \Rightarrow u$ is constant; since Γ_D was nontrivial and $u=0$ on Γ_D we get $u = 0$.

Same argument works for linear elasticity: the key integn by parts is

$$\int_{\Omega} \sum_{i,j} u_i \left(\frac{\partial}{\partial x_j}\sigma_{ij}\right) dx = \int_{\Omega} u \cdot \operatorname{div}\sigma$$

$$\parallel$$

$$\int_{\partial\Omega} \sum_{i,j} u_i \sigma_{ij} n_j \, ds - \int_{\Omega} \sum_{i,j} \frac{\partial u_i}{\partial x_j}\sigma_{ij} \, dx$$

$$\parallel$$

$$\int_{\partial\Omega} \langle u, \sigma \cdot n\rangle ds - \int_{\Omega} \langle e(u), \sigma\rangle dx$$

using symmetry of σ in last step. So: repeating argt used for Laplace's eqn gives (since $\sigma = Ae(u)$):

If u = difference of two solns then

$$\int_\Omega \langle A\, e(u), e(u)\rangle\, dx = 0$$

so (since A is positive definite).

$$e(u) \equiv 0 \text{ in } \Omega.$$

To finish, we need elastic analogue of $\nabla u = 0 \Rightarrow u =$ constant. It is:

<u>Lemma</u>: If Ω is a connected region then $e(u) \equiv 0$ in $\Omega \Rightarrow u$ is an "infinitesimal rigid motion"

$$u_i(X) = \sum_j \omega_{ij} x_j + d_i$$

for some (constant) skew-symmetric matrix ω_{ij} & some constant vector d.

Pf is easy in $\mathbb{R}^2$:

$$\partial_1 u_1 = 0 \Rightarrow u_1 = f(x_2)$$
$$\partial_2 u_2 = 0 \Rightarrow u_2 = g(x_1)$$
$$\partial_1 u_2 + \partial_2 u_1 = 0 \Rightarrow f'(x_2) + g'(x_1) = 0$$
$$\Rightarrow f = \omega x_2 + \text{const}$$
$$g = -\omega x_1 + \text{const}$$
for some $\omega \in \mathbb{R}$.

Pf in $\mathbb{R}^3$ (or $\mathbb{R}^n$) can be done similarly, using induction on n. Or, here is a different (less obvious but easier) argument: one verifies from the defn of $e(u)$ that

$$\partial^2_{jk} u_i = \partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk}.$$

so $e(u) \equiv 0 \Rightarrow \nabla\nabla u \equiv 0 \Rightarrow u$ is affine. Since $e(u)=0$, Du is skew symmetric.

What if $\Gamma_D = \phi$ (The "pure traction" bvp)?

For Laplace's eqn, we know there is a consistency condition:

$$\Delta u = f \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial n} = g \quad \text{at } \partial\Omega$$

$$\Rightarrow \int_\Omega f \, dx = \int_{\partial\Omega} g \, ds$$

since $\displaystyle\int_\Omega \Delta u = \int_\Omega \operatorname{div}(\nabla u) = \int_{\partial\Omega} \partial u/\partial n.$

To understand a little better why this

condition is needed, consider the var'l principle for the Neumann pblm:

$$\min \int_{\Omega} \tfrac{1}{2}|\nabla u|^2 + fu \, dx \; - \int_{\partial\Omega} gu \, ds$$

If $f+g$ are inconsistent then the min is $-\infty$ since we can take u = constant and drive the linear terms to $\pm\infty$.

The situation for linear elasticity is similar, except that we have a larger class of elastic-energy-free deformations:

(*) If $-\operatorname{div}\sigma = f$, $\sigma = Ae(u)$ in Ω
$\sigma \cdot n = g$ at $\partial\Omega$

has a solution then

$$\int_{\partial\Omega} \langle g, \hat{u}\rangle \, ds + \int_{\Omega} \langle f, \hat{u}\rangle \, dx = 0$$

whenever $\hat{u}$ is an infinitesimal rigid motion.

In fact: if u solves (*) and $\hat{u}$ is an infl rigid motion then

$$\int_\Omega \langle \hat{u}, \operatorname{div} \sigma \rangle + \langle \hat{u}, f \rangle \, dx = 0$$

$$\Rightarrow \int_{\partial\Omega} \langle \sigma \cdot n, \hat{u} \rangle - \int_\Omega \underbrace{\langle e(\hat{u}), \sigma \rangle}_{0} + \int_\Omega \langle \hat{u}, f \rangle = 0$$

$$\Rightarrow \int_{\partial\Omega} \langle g, \hat{u} \rangle ds + \int_\Omega \langle f, \hat{u} \rangle dx = 0.$$

<u>What about existence</u>?

For Laplace, many methods are possible, including

a) Lax-Milgram lemma (really Riesz rep'n thm since this p'bm is self-adj't)
b) var'l principle
c) maximum-principle-based methods (Perron's method)
d) bdry integral methods

For elasticity, we're dealing with a system, so <u>there is no max principle</u>. But the other 3 methods have analogues.

A full treatment of existence lies beyond the scope of this class. But let's spend some time understanding why Korn's inequality plays a crucial role.

Returning to a favorite analogue: consider the situation for Laplace's eqn with $\partial u/\partial n = 0$ at $\partial\Omega$

$$\Delta u = f \quad \text{in } \Omega$$
$$\partial u/\partial n = 0 \quad \text{at } \partial\Omega$$

and assume the consistency condition $\int_\Omega f\,dx = 0$ is met. The assoc var'l prin is

(**) $$\min_u \int_\Omega \tfrac{1}{2}|\nabla u|^2 + uf \, dx$$

How can we see that this functional is bounded below? Replacing u by u-const, we may assume wlog that $\int_\Omega u\,dx = 0$. Then

$$\int_\Omega uf \le \left(\int_\Omega |u|^2\right)^{1/2} \left(\int_\Omega |f|^2\right)^{1/2}$$

$$\le C\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} \left(\int_\Omega |f|^2\right)^{1/2}$$

using the inequality

"Poincaré's 2nd ineq" $\int_\Omega |u|^2 \le C \int_\Omega |\nabla u|^2$ when $\int_\Omega u\,dx = 0$.

It follows easily that (**) is bounded below (since $\frac{1}{2}x^2 + cx$ is bounded below as a function of $x \in \mathbb{R}$, for any choice of the constant c).

The situation for elasticity is similar. Focusing again on the special case of Neumann bc zero: for

$$-\operatorname{div}\sigma = f \text{ in } \Omega, \quad \sigma = A e(u)$$
$$\sigma \cdot n = 0 \text{ at } \partial\Omega$$

the var'l principle is

$$\min_u \int_\Omega \tfrac{1}{2}\langle A e(u), e(u)\rangle - u \cdot f \, dx$$

We must assume that the load f is orthog (in L^2) to the inf'l rigid motions. Therefore we may assume u has avg value 0 and $\int_\Omega \partial_i u_j \, dx$ is symmetric. The argument we used in the scalar case can be repeated, provided we know

"<u>Korn's 2nd ineq</u>": if $\int_\Omega \partial_i u^i \, dx = 0$ then

$$\int_\Omega |\nabla u|^2 dx \le C \int_\Omega |e(u)|^2 dx$$

for some constant $C = C_\Omega$. Indeed, then $\int_\Omega \partial_i u_j \, dx$ symmetric and $\int_\Omega u_i = 0$ for all i give

$$\left| \int_\Omega u \cdot f \right| \le \left(\int_\Omega |u|^2\right)^{1/2} \left(\int_\Omega |f|^2\right)^{1/2}$$

$$\le C \left(\int_\Omega |\nabla u|^2\right)^{1/2} \left(\int_\Omega |f|^2\right)^{1/2} \quad \text{"Poincaré's 2nd ineq"}$$

$$\le C' \left(\int_\Omega |e(u)|^2\right)^{1/2} \left(\int_\Omega |f|^2\right)^{1/2} \quad \text{by Korn's 2nd ineq.}$$

[What is Korn's 1st inequality? <u>It is</u>

$$\text{if } u\big|_{\partial\Omega} = 0 \text{ then } \int_\Omega |\nabla u|^2 dx \le C \int_\Omega |e(u)|^2 dx$$

This is what we would have needed to show the var'l pbm is bounded below when there's a Dirichlet bc $u\big|_{\partial\Omega} = 0$.].

Some intuition on Korn's 2nd ineq: if

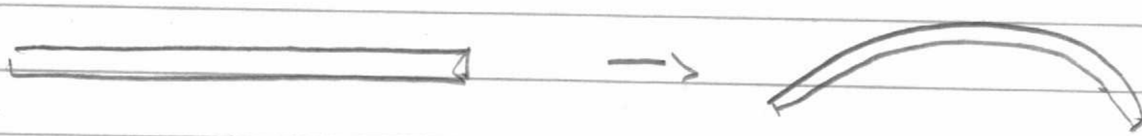
clearly implies

for any u (with no conditions), there is an inf'l rigid motion $\hat{u}$ st

$$\int_\Omega |u-\hat{u}|^2 \le C \int_\Omega |e(u)|^2$$

ie "small strain $\Rightarrow$ close to a rigid motion".

The constant depends on the domain; for a long, thin domain (like a sheet of paper) the constant is large



since deformation by bending produces small strain but u is nothing like a rigid motion.

Korn's 1st ineq is easy. Here is a proof using just integr by parts: if Ω is bounded and $u=0$ at its bdry then

$$\int_\Omega |e(u)|^2 dx = \int_\Omega \sum \left(\frac{\partial_i u_j + \partial_j u_i}{2}\right)^2$$

$$= \int_\Omega \tfrac{1}{2}|Du|^2 + \tfrac{1}{2}\sum' (\partial_i u_j)(\partial_j u_i)\, dx$$

But since $u=0$ near $\partial\Omega$, for each $i \neq j$

$$\int_\Omega (\partial_j u_i)(\partial_i u_j)\, dx = \int_\Omega -u_i\, \partial^2_{ij} u_j\, dx$$

$$= \int_\Omega (\partial_i u_i)(\partial_j u_j)\, dx$$

So

$$\int_\Omega \sum_{i,j}' (\partial_i u_j)(\partial_j u_i)\, dx = \int_\Omega (\operatorname{div} u)^2\, dx$$

whence

$$\int_\Omega |e(u)|^2 dx = \tfrac{1}{2}\int_\Omega |Du|^2 + \tfrac{1}{2}\int_\Omega (\operatorname{div} u)^2.$$

So Korn's 1st ineq says precisely that

$$u|_{\partial\Omega} = 0 \implies \int_\Omega |\nabla u|^2 dx \leq C\left[\tfrac{1}{2}\int_\Omega |\nabla u|^2 + \tfrac{1}{2}\int_\Omega (\operatorname{div} u)^2\right].$$

Evidently, it's true with $C = 2$.

Korn's 2nd ineq is more difficult.

- 1st "proof" was 1910 by Korn.

- Friedrichs wrote a paper abt 1947 pointing out importance of this result, giving a new (still not very simple) proof, & giving first modern treatment of existence thms for elasticity via Hilbert space methods

- Many proofs since Friedrichs. Some are very efficient but not so elementary (see eg the book by Duvaut + Lions that's on my syllabus). In the 60's + 70's, other "coerciveness inequalities" were considered: when does control of selected combinations of $\partial_j u_i$ yield control of the full H^1 norm? (Work by K.T. Smith, Aronszajn, + others.)

- A really simple, elementary proof was finally given by Oleinik & Kondratiev around 1989 (CRAS Paris Ser I, 1989, vol 308(16) 483-487; also Rend. Mat. Appl. (7) 10, 1990, no 3, 641-666). I'll post a handout with a version of their proof.