Mechanics - Lecture 6 - 3/1/2022

[We'll start by discussing structural restrictions on the nonlinear elastic energy density W, from pp 5.8-5.11 of the Lecture 5 notes.]

A little more about the physical implications of a lack of rank-one convexity:

- Consider first a string in tension, with reference interval $0 < s < L$, $r(0) = 0$ and an applied force $(F, 0, 0)$ at $s = L$, with $F > 0$.

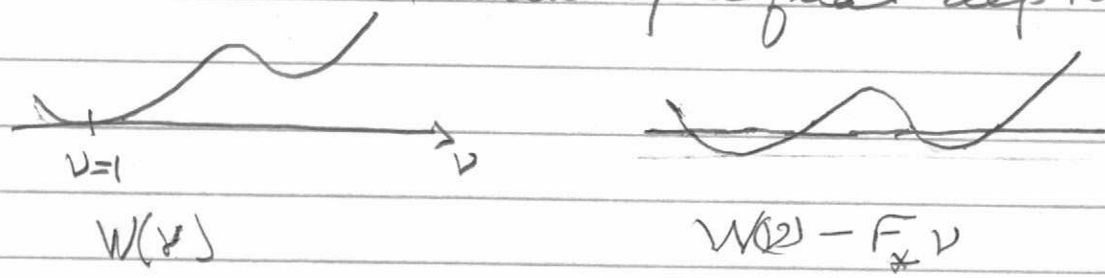
The assoc var'l problem is

$$\min \int_0^L W(|r_s|)\, ds - F r_1(L).$$

Intuition suggests $r_2 = r_3 = 0$ and $r_1' = \nu > 1$. Since $r_1(L) = \int_0^L r_1'$ this reduces the problem to

$$\min \int_0^L W(\nu(s)) - F\nu(s)\, ds$$

If the stress-strain law $\nu \to N(\nu) = W'(\nu)$ is not monotone for $\nu > 1$ then for some F, $\nu \to W(\nu) - F\nu$ will have 2 local min of equal depth



For $F<F_*$ the left min is preferred; for $F>F_*$ the right min is preferred; for $F=F_*$ either one is OK, <u>and so is any mixture</u>. (Mixtures are possible since a piecewise constant ν integrates to a piecewise affine $\vec{r}(s)$.)

In HW1, this situation occurred in problem 3(d) [the uniform ring under pressure, if the stress-strain law is non monotone]. In HW3, something similar is at the heart of problem 1(d),(e), except that there it's the isotropic stretch λ that plays the role of ν.

- <u>In 3D nonlinear elasticity</u> we can consider a similar setup with energy

$$\int_\Omega W(Du)\,dx - \int_{\partial\Omega} \sum u_j G_{j\alpha} N_\alpha \, ds$$

where G is a constant matrix (physically, this means we apply force $G \cdot N$ per unit reference area at $\partial\Omega$). Since this is equivalent to

$$\int_{\Omega} W(Du) - \sum G_{i\alpha} \frac{\partial u_i}{\partial X_\alpha} \quad dX$$

we are in a situation very much like the 1st bullet.

But: the higher-dimensional setting is different from 1D, since if u is cont's then only special jumps of Du are permitted

$$Du = A \,\Big|\, Du = B \quad \Rightarrow \quad (B-A)\cdot \vec{t} = 0$$
$$\rightarrow \vec{n} \qquad\qquad \text{whenever } \vec{t} \perp \vec{n}$$

$$\Rightarrow B - A = \vec{a} \otimes \vec{n}$$
for some vector $\vec{a}$

If W is (strictly) rank-one convex then the phenomenon seen in the 1D setting (with a nonmonotone stress strain law) cannot occur.

Now we turn to <u>linear elasticity</u>. This entails <u>two</u> distinct linearizations:

<u>1st</u> <u>geometrical linearization</u> ("small strain theory") – we suppose

$$x_\delta(X) = X + \delta u(X)$$

with δ small. [Note that we work near the <u>identity</u> – not near an arbitrary rotn.] Here u is the "infinitesimal elastic displacement." Clearly

$$F = I + \delta \cdot Du$$

$$(F^T F)^{1/2} = I + \frac{\delta}{2}(Du + Du^T) + O(\delta^2)$$

so to leading order $(F^TF)^{1/2} - I$ is δ times

$$e(u) = \frac{1}{2}(Du + Du^T)$$

(the "linear elastic strain").

<u>2nd</u> <u>physical linearization</u> ("linear stress-strain law") – using hyperelasticity as a starting pt & with $W(I) = 0$

and $W > 0$ except at rotations, we have

$$W(F) = \delta^2 < \alpha e(u), e(u) > + O(\delta^3)$$

where $<\alpha e, e>$ is the quadratic form assoc with the Hessian of W. Evidently (assuming some nondegeneracy)

$$<\alpha e, e> = \sum{}' \alpha_{ijkl}\, e_{ij}\, e_{kl}.$$

is a positive definite (symmetric) quadratic form on symmetric matrices.

Dropping terms of higher order in δ (and rescaling by δ^2) we expect a var'l prin of the form $\delta E = 0$ where

$$E = \int_\Omega \tfrac{1}{2} <\alpha e(u), e(u)> dx + \left[\begin{array}{l}\text{terms assoc}\\ \text{to loads}\end{array}\right]$$

so equil eqn is

$$\operatorname{div}(\alpha e(u)) + f = 0$$

and we recognize

$$\sigma = \alpha\, e(u)$$

as the stress tensor in the linearly elastic setting. We call α its "Hooke's law".

<u>Notes</u>:

a) σ is symmetric

b) there is no distinction between Cauchy + Piola-Kirchhoff stress in linear elasticity. (Why not? Because we're working locally near $x(X) = X$. The reference + deformed volume + area elements agree at leading order in δ.)

<u>Summary</u>: in linear elasticity the basic unknown is the elastic displacement $u(x)$. Assoc lin strain is $e(u) = \frac{1}{2}(Du + Du^T)$. Stress is $\sigma = \alpha\, e(u)$. Eqn of equil is

$$\operatorname{div} \sigma + f = 0 \quad , \quad \sigma = \alpha\, e(u)$$

(f = force/unit vol = "body load"), augmented by suitable bc such as

- $u = u_0$ at $\partial\Omega$ ("displacement bc")

or

- $\sigma \cdot n = g$ at $\partial\Omega$ ("traction bc")

or

- $u \cdot n = 0$, $(\sigma \cdot n)_{\tan} = 0$ at $\partial\Omega$ ("lubricated bc").

We discussed isotropy in the nonlinear setting. The analogue for linear elasticity is

Hooke's law α is isotropic $\Leftrightarrow$ $\alpha(R^T e R) = R^T(\alpha \cdot e) R$. for any (symmetric) e and any (orientation preserving) rotn R

(More generally: a rotn R is a symmetry of the material if $\alpha(R^T e R) = R^T(\alpha \cdot e) R$. for any symmetric e.)

<u>Lemma</u>: The general isotropic Hooke's law can be expressed as

$$\sigma_{ij} = 2\mu\, e_{ij} + \lambda (\operatorname{tr} e)\, \delta_{ij}$$

where λ & μ are constants (Lamé moduli).

The proof is in every textbook. Rather than prove it here, let me suggest <u>why</u> it's true:

Observe that Hooke's law is isotropic $\Longleftrightarrow$ $\langle \alpha e, e \rangle$ is a quadratic, isotropic fn of symmetric tensors e (that is: it has the same value at e and at $R^T e R$.)

Fact of tensor algebra: a basis for such fns is obtained by starting from

$$e_{ij}\, e_{kl}$$

and "contracting indices in pairs." There are two ways to do this:

- $\sum_{i,k} e_{ii}\, e_{kk} = (\operatorname{tr} e)^2$
- $\sum_{i,j} e_{ij}\, e_{ij} = \operatorname{tr}(e^2) = |e|^2$

So isotropy $\Longrightarrow$

$$\langle \alpha e, e \rangle = 2\mu |e|^2 + \lambda (\operatorname{tr} e)^2$$

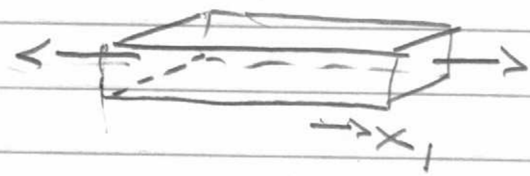
for some λ and μ.

There are other useful reps of an isotropic Hooke's law, with clearer physical interpretations:

- one uses Young's modulus E and Poisson's ratio ν

$$\sigma_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}(\mathrm{tr}\, e)\,\delta_{ij}\right)$$

To see meaning of E & ν, consider behavior under uniaxial tension



$\sigma_{11} = T$, other components of $\sigma = 0$

$$\Rightarrow e_{11} = \frac{1}{E} T$$

$$e_{22} = e_{33} = -\frac{\nu}{E} T$$

Most materials have $\nu > 0$, so uniaxial tension produces contraction in the orthog variables (and uniaxial compression produces expansion). Cork is different: it has $\nu \approx 0$, which is why it is useful for closing wine bottles

- bulk and shear moduli describe

how α behaves on the two rotation-invariant subspaces

$$\{\text{multiples of } I\} \text{ and } \{e : \operatorname{tr} e = 0\};$$

specifically:

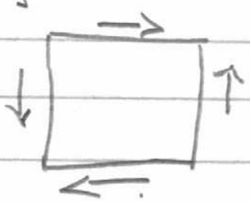
$$\sigma_{ij} = 3K \cdot \underbrace{\tfrac{1}{3}(\operatorname{tr} e)\,\delta_{ij}}_{\substack{\text{proj'n of}\\ e \text{ to multiples}\\ \text{of } I.}} + 2\mu \underbrace{\left(e_{ij} - \tfrac{1}{3}(\operatorname{tr} e)\,\delta_{ij}\right)}_{\substack{\text{proj'n of } e\\ \text{orthog to } I.}}.$$

Physical interpretation:

- bulk modulus controls how volume changes in response to hydrostatic pressure

- shear modulus describes response to "pure shear load" eg $\sigma_{12} = T$, $\sigma_{ij} = 0$ otherwise

2D picture in $x_1 x_2$ plane



loads.

$$\Rightarrow \quad e_{12} = \frac{1}{2\mu}\sigma_{12} \quad (e_{ij} = 0 \text{ otherwise})$$

in 2D, this example gives $Du = \frac{1}{2\mu}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (one diagonal is stretched, the other is shrunk)

[Caution: in 2D, proj'n of e to multiples of I is $\frac{1}{2}(\mathrm{tr}\, e)I$.]

Of course these different expressions for α are related:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$\lambda = K - \frac{2}{3}\mu$$

Positivity of $\langle \alpha e, e \rangle$ requires

$\mu > 0$	$E > 0$	$\mu > 0$
$3\lambda + 2\mu > 0$	$-1 < \nu < 1/2$	$K > 0$
for Lamé moduli	for Young's modulus + Poisson's ratio	for bulk + shear moduli.

For analysis (and numerics), var'l principles are very important. Simplest examples:

<u>Dir bc</u> φ

$$\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle - \langle u, f\rangle \, dx$$

<u>Traction bc</u> ($\sigma \cdot n = g$ at $\partial\Omega$)

$$\min_u \left\{ \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle - \langle u, f\rangle \, dx - \int_{\partial\Omega} \langle u, g\rangle \, ds. \right\}$$

Both are convex – but degenerate since the quadratic term sees only the <u>symmetric part</u> of Du. So it is natural to ask:

Q1) if $\int_\Omega |e(u)|^2 < \infty$, is $u \in H^1(\Omega)$?

Q2) if $e(u) \equiv 0$, must u be an "infinitesimal rigid motion" ($u = M \cdot x + b$ with M a <u>constant</u> skew-symmetric matrix)?

Korn's inequality gives the answers (and more). We'll discuss it next week.