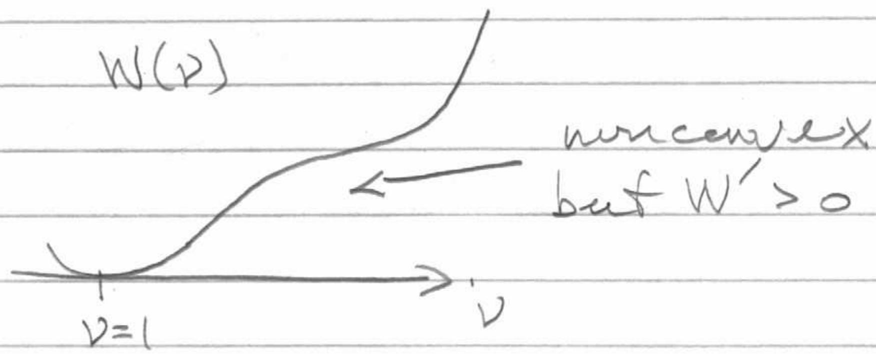
Addenda to the Lecture 6 notes 3/3/2022

(1) The 1st figure on pg 6.2 (sketching $W(\nu)$) was not right. For a string in tension ($\nu > 1$) we expect $N(\nu) > 0$, even if $N = W'$ is nonmonotone (W is nonconvex). So the graph of W should look (in the nonmonotone stress-strain law case) like



(2) I wasn't very clear about how the nonlinear energy density determines the Hooke's law. Recall that in nonlinear elasticity the energy density W should be frame indifferent

$$W(F) = W(RF)$$ for any orientation-pres rotation R

Writing any F (with $\det F > 0$) as

$$F = R\,(F^T F)^{1/2}$$ (polar decomposition)

we see that W is fully determined by its restriction to symmetric matrices; let's call that

$$\Phi = \text{restriction of } W \text{ to pos def symmetric matrices.}$$

Φ has a local min at the identity so its Taylor expansion at I has the form

$$\Phi(I + \delta e) = \tfrac{1}{2}\delta^2 \langle D^2\Phi(I)\, e, e\rangle + O(\delta^3)$$

for any symmetric matrix e. Taking $e = e(u)$, we see that

$$\text{the Hooke's law of linear elasticity} = D^2\Phi(I)$$

ie the linear elastic stress is $\sigma_{ij} = \sum A_{ijkl}\, e_{kl}$ where

$$A_{ijkl} = \frac{\partial^2 \Phi}{\partial E_{ij}\, \partial E_{kl}}(I).$$

where $\Phi(E) = W(E)$ = restriction of W to symmetric matrices E.