Mechanics – Lecture 5 – 2/22/2022

[We'll start by finishing the Lecture 4 notes.]

We turn now to constitutive laws for nonlinear elasticity.

Two viewpts are possible:

(a) "hyperelasticity": assume there's an underlying var'l principle, in other words specify the Piola-Kirchhoff stress

$$P_{i\alpha} = \frac{\partial}{\partial F_{i\alpha}} W(X, F)$$

where W = "elastic energy density" at reference pt X and def gradient F. [This is the viewpt introduced in Lecture 4.]

(b) "Cauchy elasticity": specify Cauchy stress $\tau = \hat{\tau}(X, F)$ as a fn of reference position X and def gradient F

Often a body might be homogeneous; then $W = W(F)$ & $\hat{\tau} = \hat{\tau}(F)$.

Clearly (b) is more general than (a). [In a hyperelastic material there *is* a rule of the form (b); but not every $\hat{\Sigma}$ comes from an elastic energy [this will be easier to see when we get to linear elasticity].]

There *are* settings where the "Cauchy elastic" viewpt is needed (see recent papers by Vincenzo + others on "odd elasticity".) But here I'll stick to the *hyperelastic* setting.

We *always* require "frame indifference". For hyperelasticity, this says

(*) $$W(RF) = W(F)$$ for any orientation-preserving rotation R.

For Cauchy elasticity it says

(**) $$\hat{\Sigma}(RF) = R\,\hat{\Sigma}(F)\,R^T$$ for any orientation-pres rotation R.

[You'll be asked on HW3 to check that when $\hat{\Sigma}$ comes from a frame-indifferent W it does satisfy (**).]

Interpretation of (*) : rotation does no elastic work; interp. of (**) : an observer in a rotated coord system sees the same basic stress-strain law.

Let's check that when the Piola-Kirchhoff stress comes from an elastic energy

$$P_{i\alpha} = \frac{\partial W}{\partial F_{i\alpha}} \quad \text{with } W \text{ frame-indifferent}$$

the associated Cauchy stress Σ is at least symmetric. In fact, recalling that

$$P = J\,\Sigma(F^{-1})^T \quad \left(\text{so } \Sigma = J^{-1} P F^T\right)$$

with $J = \det F$, our task is to show that

$$\text{frame indifference of } W \Rightarrow PF^T = \sum_\alpha P_{i\alpha} F_{j\alpha} \text{ is symmetric.}$$

For any skew-symmetric matrix Ω_{ij} there's a rotation-valued curve $R(t)$ st $R(0) = I$ and $\dot{R}(0) = \Omega$. Frame indifference gives

$$0 = \frac{d}{dt}\Big|_{t=0} W(R(t)F) = \sum_{i,\alpha}' P_{i\alpha}(F)(\dot{R}F)_{i\alpha}$$

$$= \sum_{i,\alpha,j}' P_{i\alpha}(F)\Omega_{ij}F_{j\alpha}$$

So $P(F)F^T$ is orthogonal to the space of skew-symmetric matrices. So it is symmetric.

Many materials are <u>isotropic</u>. In hyperelasticity

isotropy $\iff$ $W(F) = W(FR)$ for any rotn R.

$\iff$ $W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ where $\{\lambda_i\}$ are the principal stretches and φ is a <u>symmetric</u> function of its arguments

Corresp assertion for Cauchy elasticity:

isotropy $\iff$ $\hat{\Sigma}(FR) = \hat{\Sigma}(R)$ for any rotn R.

There are some <u>structural requirements</u> that an elastic energy should satisfy. The most basic is that

a) eqns of elastostatics are elliptic (and eqns of elastodynamics are hyperbolic).

A somewhat stronger condition is

b) var'l principle of elastostatics should achieve its min energy.

Discn of (b) would take us too far afield, so we'll stick to (a). [But let me mention: (b) $\Leftrightarrow$ the elastic energy density is "quasiconvex".]

Essence of (a) is that $F \to W(F)$ should be (strictly) <u>rank-one convex</u>

$$\sum \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}} \xi_i \xi_j \eta_\alpha \eta_\beta \geq C|\xi|^2|\eta|^2$$

for all $\xi, \eta \in \mathbb{R}^3$. (Equivalently: restn of W to a rank-one line $F + t\xi\otimes\eta$ is a convex fn of t.)

Intuition about why this should be needed: ellipticity is assessed by "freezing the coeffts" in the highest-order term, which for $\mathrm{div}_x P$ is $\sum' \limits_{\alpha,\beta,j} \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}} \frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta}$,

so let's discuss the pde

$$\sum{}' A_{i\alpha j\beta} \frac{\partial^2 u_j}{\partial X_\alpha \partial X_\beta} = f_i$$

with $A_{i\alpha j\beta}$ constant. If the pde holds in all $\mathbb{R}^n$ then we can use Fourier transform:

$$-\sum_{\alpha,\beta,j}{}' k_\alpha k_\beta A_{i\alpha j\beta} \hat{u}_j(k) = \hat{f}_i(k)$$

The condition

$$\sum_{\alpha,\beta,i,j}{}' A_{i\alpha j\beta} \xi_i \xi_j \eta_\alpha \eta_\beta \geq C|\xi|^2|\eta|^2$$

assures that $\sum\limits_{\alpha,\beta} k_\alpha k_\beta A_{i\alpha j\beta}$ (which must be inverted) is positive definite.

Why not just take W to be convex? Well, the set of all rotations $SO(3)$ is <u>not</u> convex. So

W is min at $F \in SO(3)$ [only] $\Rightarrow$ W cannot be convex

More concretely: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_0$ is an orientation-preserving rotation; but if $W(I) = 0$ + $W(R_0) = 0$ then $W(\frac{1}{2}I + \frac{1}{2}R_0) \leq 0$, yet $\frac{1}{2}I + \frac{1}{2}R_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ isn't even invertible.

Where to look for acceptable W's?

<u>Fact</u>: if $W(F) =$ convex fn of F
+ convex fn of $\det F$

then W is rank-one convex (indeed, even "quasiconvex").

How to find a frame-indifferent convex fn of F? Use this result (proved eg as Thm 4.9-1 in Ciarlet's book): if $W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ in terms of λ_i = principal stretches (the eigenvalues of $(F^TF)^{1/2}$) with φ symmetric, convex, and nondecreasing in each variable, then W is a convex fn of F.

A simple (though extreme) case is the "<u>incompressible</u> <u>neoHookean</u>" material

$$W(F) = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad \text{restr by } \lambda_1\lambda_2\lambda_3 = 1:$$

(Note: $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathrm{tr}(F^T F) = \sum F_{i\alpha}^2$ is obviously convex!)

There's much more to say about where to find choices of W that are rank-one convex & also frame-indifferent and minimized exactly at the rotations. Ciarlet's chap 4 has a good treatment.

We briefly touched on incompressible elasticity above. In fact rubber and some other polymers are very nearly incompressible so the neoHookean law is widely used (at least when the strain is small).

But what does the pde of static equilibrium look like in this case? Short answer: it includes an unknown function – the pressure – as a Lagrange multiplier for the constraint of incompressibility; the constitutive law is in fact

$$\tau = -p I + \tau^*(F)$$

where τ^* is given by an elastic energy using the methods discussed above (but applied only when $\det F = 1$) and p is determined by balance of forces.

<u>Explain this</u>: starting from constrained var'l principle:

$$0 = \delta_{\det(\partial x/\partial X)=1} \int_\Omega W(\partial x/\partial X)\, dX,$$

by method of Lagrange mult the EL eqn is formally

$$0 = \delta \int_\Omega W(\partial x/\partial X) + \gamma(X)\left[\det\left(\frac{\partial x}{\partial X}\right) - 1\right] dX$$

for some (unknown) $\gamma(X)$. Our task is to show that the Cauchy stress assoc to $\gamma(X)\left[\det\left(\frac{\partial x}{\partial X}\right)-1\right]$ has the form $-pI$ for some $p(x)$. The key to this is Cramer's rule, which says

$$\frac{\partial(\det F)}{\partial F_{i\alpha}} = J(F^{-1})^T$$

(Note: LHS is the matrix of minors!)
Recalling that $P = J\,\tau\,(F^{-1})^T$, ie $\tau = J^{-1} P F^T$,

we get
$$P = \gamma \frac{\partial \det F}{\partial F_{i\alpha}} \Rightarrow \Sigma = J^{-1}(\gamma J (F^{-1})^T) F^T = \gamma I.$$
so the assertion is proved (with $\gamma(X) = -p(x)$).

Returning to the general (compressible) case, let me mention another viewpt that's sometimes useful.

I said before that in the isotropic case $W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ where λ_i are the principal stretches + φ is a symmetric fn of 3 variables. A different representation of the same set of fns is

$$W(F) = \Psi(\mathrm{I}, \mathrm{II}, \mathrm{III})$$

where $\mathrm{I}, \mathrm{II}, \mathrm{III}$ are the elementary symmetric fns of the eigenvalues of $F^TF = C$, ie

$$\mathrm{I} = \operatorname{tr} C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$\mathrm{II} = \tfrac{1}{2}[(\operatorname{tr} C)^2 - \operatorname{tr} C^2] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$$
$$\mathrm{III} = \det C = \lambda_1^2\lambda_2^2\lambda_3^2$$

Key advantages of this repn: Ψ has no symmetry reqts, and the Piola-Kirchhoff stress is polynomial expression in derivs of φ and components of F.

There is an analogous framework for constitutive modeling in Cauchy elasticity:

$$T(F) = \varphi_0 I + \varphi_1 B + \varphi_2 B^2$$

where $B = FF^T$ and φ_i are suitable fns of I, II, III. (Note: $B = FF^T$ and $C = F^TF$ have the same eigenvalues, though their eigenvectors are generally different.) HW 3 will ask you to identify φ_0, φ_1, and φ_2 when $W(F) = \psi(I, II, III)$.

HW 3 will also explore

- how nonlinearities of elasticity explain observable effects like reln b'tw pressure & radius when blowing up a (spherical) balloon

- how expts can be used to identify the constitutive law.

(Lecture 6 will start linear elasticity.)