Mechanics - Lecture 4, 2/15/2022

Today's topic: nonlinear elasticity

We started at end of Lecture 3 with discn of strain. Reminder of key pt: if $F_{i\alpha} = \partial x_i/\partial X_\alpha$ is def gradient then its polar decomposition

$$F = RU \qquad U = (F^T F)^{1/2}$$

carries the essential info: eigenvalues of U are "principal stretches", and F is an isometry (no strain) iff $U = I$.

What about stress? (Discussion of forces; "statics"; Cauchy's thm is closely linked to the fact that the natural objects to integrate over 2D surfaces are 2-forms.)

We consider 2 types of forces:

- a body force $\vec{f}(x)$ per unit deformed volume, acting at (deformed) position x
- the surface force $\vec{T}(x;\vec{n})$ per unit

deformed area, acting on plane $\perp$ unit vector $\vec{n}$ at (deformed) position x. (Sign convention: $\vec{T} = -p\vec{n}$ with $p > 0$ for hydrostatic pressure.)

Intuition: a spatial volume is acted upon by body force (eg gravity) and also by surface forces (applied by the rest of the body).

Note convention: $\vec{f}$ = force/unit deformed vol. Assoc force per unit reference vol is $f \cdot \det\left(\frac{\partial x}{\partial X}\right)$. [In lectures 1+2, $\vec{f}$ was the force per unit reference length; that was a different convention].

Also note that

$$\vec{T}(\vec{n}) = -\vec{T}(-\vec{n})$$

("law of equal reaction").

Less obvious, but very important, is:

Cauchy's Theorem:

(a) $\vec{T}(x, \vec{n})$ is linear in $\vec{n}$, i.e.

$$T_i(x, \vec{n}) = \sum_{j=1}^{3} T_{ij}(x)\, n_j$$

(b) the matrix $T_{ij}(x)$ is symmetric, i.e.

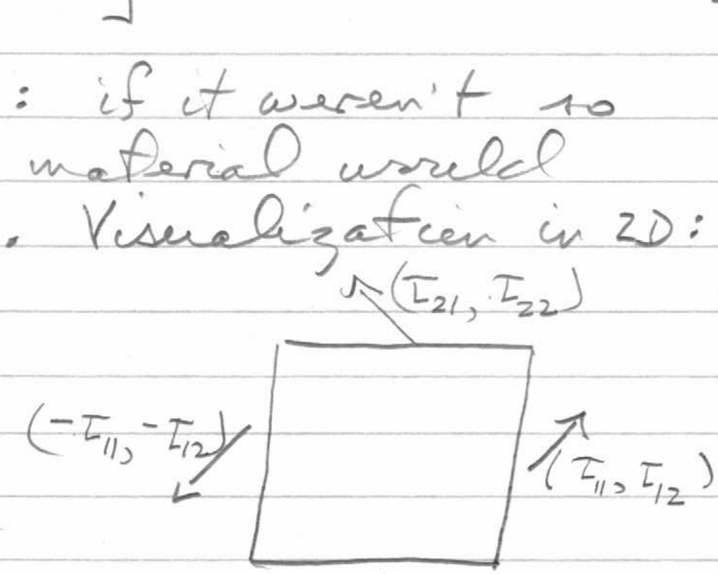
$$T_{ij} = T_{ji} \quad \text{for all } i \neq j$$

We call T the <u>Cauchy</u> <u>stress</u> <u>tensor</u>. (In elastodynamics, $\vec{T} = \vec{T}(x, t; \vec{n})$; the same result holds, with $T = T_{ij}(x,t)$.)

<u>Intuition behind (a)</u>: it says, essentially, that stress is a 2-form. (Proof resembles discn in H. Whitney's book "Geometric Integration Theory" — 1st few pages only! — why 2-forms are the natural objects to integrate over surfaces.)

<u>Intuition behind (b)</u>: if it weren't so then a small cube of material would experience a net torque. Visualization in 2D:

if $T_{12} \neq T_{21}$ then forces on faces could be as shown (figure has $T_{21} < 0$, $T_{12} > 0$)



Pf of Cauchy's thm, assuming everything is in static equilibrium [dynamic case is only slightly different, see note below].

Pf of (a): Fix x_0 + $\vec{n}$; consider tetrahedron (in deformed coords) V with corner x_0 + 3 faces $\perp$ axes (call them S_1, S_2, S_3) + 4^{th} face $\perp \vec{n}$ (call it S_0).

the case $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$

Geometry $\Rightarrow |S_i| = n_i |S_0|$ if $n_i > 0$

Outward normal to S_i is $-e_i$

Balance of forces $\Rightarrow$

$$0 = \int_{S_0} T(x,\vec{n})\, dA + \sum_{i=1}^{3} \int_{S_i} T(x,-e_i)\, dA + \int_V f\, d\mathrm{vol}$$

Assuming a little regularity, divide by $|S_0|$ and take small-vol limit to get

$$T(x,\vec{n}) + \sum_{i=1}^{3} n_i T(x,-e_i) = 0$$

(Vol integral scales like vol not area, so it disappears in limit). Thus

$$T(x, \vec{n}) = \sum_{j=1}^{3} T_{ij}(x)\, n_j$$

with $T_{ij} = T_i(x, e_j)$.

[Proof in dynamic setting is almost the same; sole difference: there's an additional volume integral $\int \rho x_{tt}\, d\mathrm{vol}$, which disappears in the limit as the bdy force did.]

We also see the equilibrium pde at this stage:

$$(*) \qquad \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} + f_i = 0$$

since total force on any volume should vanish.

<u>Proof of (b)</u>: part (a) came from balance of forces; part (b) comes from <u>balance of torque</u>. In static setting

$$\int_V x \wedge f \, d\mathrm{vol} + \int_{\partial V} x \wedge (T \cdot n)\, dA = 0$$

where $a \wedge b$ is the cross product of $a, b \in \mathbb{R}^3$

$$(a \wedge b)_i = \sum_{j,k}' \varepsilon_{ijk} a_j b_k = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

PDE version of balance of torque is

$$\sum_{j,k}' \varepsilon_{ijk} x_j f_k + \sum_{j,k,l}' \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j T_{kl} \right) = 0.$$

Simplify using balance of forces (*) to get

$$\sum_{j,k,l}' \varepsilon_{ijk} \delta_{jl} T_{kl} = 0 \quad \text{for each } i$$

ie

$$\sum_{j,k}' \varepsilon_{ijk} T_{kj} = 0 \quad \text{for each } i$$

whence T is symmetric.

[Here is a less coordinate-bound version of the same proof: balance of torque says

$$(**) \quad \int_{\partial V} x_i T_j(n) - x_j T_i(n) \, dA + \int_V x_i f_j - x_j f_i \, d\mathrm{vol} = 0$$

for each $i \neq j$ and any region V. But

$$\int_{\partial V} x_i T_j(n) - x_j T_i(n)\, dA$$

$$= \int_{\partial V} \sum_k (x_i T_{jk} - x_j T_{ik}) n_k \, dA$$

$$= \int_V \sum_k \frac{\partial}{\partial x_k} (x_i T_{jk} - x_j T_{ik})\, d\mathrm{vol}$$

$$= \int_V [T_{ji} + x_i (\operatorname{div} T)_j$$

$$- T_{ij} - x_j (\operatorname{div} T)_i]\, d\mathrm{vol}.$$

$$= \int_V (T_{ji} - T_{ij}) - (x_i f_j - x_j f_i)\, d\mathrm{vol}$$

So (**) becomes $\int_V (T_{ji} - T_{ij})\, d\mathrm{vol} = 0$.

True for any $V \Rightarrow T_{ij} = T_{ji}$.]

[As for part (a), dynamical version is not really different.]

Done with stress + strain. Also with balance laws (at least for statics).

What's left? We need to discuss

- constitutive laws (reln btwn stress + strain)

- "reference" vs "deformed" (Lagrangian vs Eulerian) coordinates

- examples

These notes do a simple first pass at constitutive laws, then focus on the reln btwn reference + deformed coords. (We'll do more on constitutive laws in Lecture 5; you'll do some examples in HW 3).

Constitutive laws + reference vs deformed coords are intertwined, since easiest approach to constitutive laws is via variational principles (which work best in reference coords), though measurements and most practical loading devices (eg pressure) are best described in deformed coords.

<u>1st pass at constitutive law</u>: 1st variation of $E = 0 \iff$ elastostatic equilibrium, where

$$E = \int_\Omega W(\partial x/\partial X)\, dX + \int_\Omega V(x(X))\, dX$$

where $W(F)$ is a fn of matrices ("elastic energy density") satisfying certain structural conditions (to be discussed later, but at least we expect

$$W(RF) = W(F) \quad \text{for any } R \in SO(3)$$

("frame indifference") and

$$W \geq 0 \text{ with } W(R) = 0 \text{ for } R \in SO(3)$$

(if the reference config has strain 0).

The 1st varn of E vanishes $\iff$

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial X_\alpha} \left[\frac{\partial W}{\partial F_{i\alpha}} (\partial x/\partial X) \right] - \frac{\partial V}{\partial x_i}(x(X)) = 0$$

for each i. This expresses balance of forces <u>in reference coords</u>:

$\frac{\partial W}{\partial F_{i\alpha}}\left(\frac{\partial x}{\partial X}\right)$ = force in dir'n i per unit <u>reference area</u>, acting on surface $\perp$ α^{th} coord vector in ref coords

= (by defn) "1st Piola-Kirchhoff stress tensor" $P_{i\alpha}$

$-\frac{\partial V}{\partial x_i}(x(X))$ = force per unit <u>ref</u> vol acting at location $x(X)$

This viewpt is convenient because it gives a var'l principle and a pde on a <u>known</u> domain Ω.

Key to translation between Eulerian + Lagrangian versions: evidently the Cauchy stress τ_{ij} + body force per unit deformed vol f_i must satisfy

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\tau_{ij}) + f_i = 0$$

$$\Longleftrightarrow \quad \sum_{\alpha=1}^{3} \frac{\partial}{\partial X_\alpha}(P_{i\alpha}) + f_i^R = 0$$

where $f_i^R = -\frac{\partial V}{\partial x_i}(x(X))$ is the force per unit ref volume.

The correspondence btwn f + f^R is clear: we want

$$f^R \, d\mathrm{vol}_X = f \, d\mathrm{vol}_x$$

so

$$f^R = f \det\left(\frac{\partial x}{\partial X}\right).$$

The correspondence btwn $P_{i\alpha}$ and T_{ij} is also "just calculus", though we must work a bit more:

<u>Claim</u>: $P_{i\alpha} = \det\left(\frac{\partial x}{\partial X}\right) \cdot \sum_k' T_{ik} \frac{\partial X_\alpha}{\partial x_k}$

(in matrix notation: $P = J T (F^{-1})^T$ where $F_{i\alpha} = \partial x_i / \partial X_\alpha$ and $J = \det F$)

<u>Pf of claim</u>:

$$\sum_j' \frac{\partial}{\partial x_j}(T_{ij}) + f_i = 0 \iff 0 = \int -\sum_j' T_{ij} \frac{\partial \varphi}{\partial x_j} + f_i \varphi \, dx$$

for every compactly supported φ (here i is fixed). Change to X variables:

$$0 = \int \left[- \sum' \tau_{ij} \frac{\partial \varphi}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial x_j} + f_i \varphi \right] \det\left(\frac{\partial x}{\partial X}\right) dX$$

for all φ with cpt supt. This is equiv to

$$\frac{\partial}{\partial X_\alpha} \left[\sum_j' \tau_{ij} \frac{\partial X_\alpha}{\partial x_j} \det \frac{\partial x}{\partial X} \right] + f_i \det\left(\frac{\partial x}{\partial X}\right) = 0 .$$

which proves the claim.

<u>Notes</u>:

- The Eulerian stress τ_{ij} is symmetric, but the Piola-Kirchhoff stress $P_{i\alpha}$ is <u>not</u>.
- When studying fluid-solid interaction it is natural to consider "constant pressure p_0" as a bdry condition. This means

$$\tau \cdot \vec{n} = - p_0 \vec{n} \quad \text{at } \partial x(\Omega)$$

where $\vec{n}$ = unit normal to $\partial x(\Omega)$.

If $p_0 = 0$ then the bdry is traction-free and this is equivalent to $P \cdot N = 0$ ($P \cdot N = \sum_\alpha P_{i\alpha} N_\alpha$, N = unit normal to $\partial\Omega$).

However if $p_0 \neq 0$ the assoc condn on P is more complicated (it can be found by arguing as in our proof of the claim).

A fluid can be viewed as an elastic solid that resists only change of volume (no resistance to shear); this will be a HW problem.

We usually do fluid dynamics in Eulerian coords, appealing to cons of mass + momentum to get the pde's. So to compare with fluid dynamics, we need to think abt reln btwn Lagrangean + Eulerian variables.

<u>Eqns of elastodynamics</u> in Lagrangean variables:

$$\sum_\alpha \frac{\partial}{\partial X_\alpha}(P_{i\alpha}) + m(X)g = m(X)\,\ddot{x}_i$$

where

$m(X)$ = mass per unit ref vol.
$m(X)g$ = body force per unit ref vol
$\ddot{x}$ = 2nd deriv in t, holding ref position fixed

To recognize cons of mass + momentum in their usual fluid-dynamic forms, we should write these in Eulerian coords. It's important to distinguish

$$\frac{Df}{Dt} = \text{deriv of } f \text{ wrt } t, \text{ holding } X \text{ fixed}$$

$$\frac{\partial f}{\partial t} = \text{deriv of } f \text{ wrt } t, \text{ holding } x \text{ fixed.}$$

By chain rule,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \sum v_i \cdot \frac{\partial f}{\partial x_i} \quad \text{where } v_i = \frac{Dx_i}{Dt}$$

(on pg 4.13, we wrote $\dot{x}$ for Dx_i/Dt).

<u>Claim</u>: Eqns of elastodynamics are equiv to

$$\frac{\partial \rho}{\partial t} + \sum_j \frac{\partial}{\partial x_j}(\rho v_j) = 0 \qquad \text{[cons of mass]}$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v \cdot \nabla v_i\right) = \sum_j \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i \qquad \text{[cons of momentum]}$$

where $\rho(x,t) = m(X(x,t))\det\left(\frac{\partial X}{\partial x}\right)$ = mass per unit deformed volume.

Proof: For 1st eqn, we have

$$0 = \frac{d}{dt}\int_B m\, dX \quad \text{for any } B \subset \Omega$$

$$= \frac{d}{dt}\int_{x(B,t)} \rho\, dx$$

$$= \int_{x(B,t)} \frac{\partial \rho}{\partial t} \quad \int_{\partial x(B,t)} \rho\, v \cdot n$$

$$= \int_{x(B,t)} \left[\frac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho v)\right] d\mathrm{vol}.$$

True for all $B \Rightarrow$ 1st eqn. Now for 2nd eqn recall that $\operatorname{div}_X P = J \operatorname{div}_x T$, with $J = \det(\partial x/\partial X)$. So

$$m\ddot{x} = \operatorname{div}_X P + mg \iff J^{-1} m\ddot{x} = J^{-1}\operatorname{div}_X P + J^{-1} m g$$

$$\iff \rho\ddot{x} = \operatorname{div}_x T + \rho g$$

using defn of ρ. Now $\ddot{x} = \dot{v} = \frac{Dv}{Dt} = v_t + v\cdot\nabla v$ so we get the second eqn (cons of momentum).