Mechanics – Lecture 3, 2/8/2022

Main topic today: an intro to bifurcation, in the context of Euler's elastica (continuing from the last part of Lecture 2).

If time permits, we'll finish by laying some groundwork for discussion of 3D elasticity.

Recall from Lecture 2: we focus on the elastica with horizontal compressive load λ at RHS and a clamped bdry $\theta(0)=0$ at LHS; also we take $L=1$!

$0<s<1$ reference $\qquad\qquad$ $\theta(0)=0$ $\quad (\vec{r}(0)=(0,0))$

The eqn is thus

$$(*) \qquad A\theta'' + \lambda \sin\theta(s) = 0 \qquad 0<s<1$$
$$\theta(0)=0, \quad \theta'(1)=0$$

We discussed a variational perspective in Lecture 2 (and that perspective is discussed further in HW2). Here we'll explore a different approach, seeking

$$\theta = \Theta(\lambda, s)$$

as a function of both λ and s.

<u>Reminder about implicit function theorem:</u>

if

$$F(\lambda, x) = y \quad \text{maps } \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$$

and

$$F(0, x_0) = y_0$$

and

$$\left[\frac{\partial F_i}{\partial x_j}(x_0)\right]_{\substack{1\le i\le N\\ 1\le j\le N}} = D_x F(x_0) \text{ is invertible}$$

then the eqn $F(\lambda, g(\lambda)) = y_0$ determines a smooth function $g(\lambda)$ which is the unique soln of $F(\lambda, x) = y_0$ near $\lambda = 0$, $x = x_0$. Formal "proof" is easy: diff'n of eqn in λ gives

$$\partial_\lambda F + D_x F \cdot g' = 0$$

which permits us to solve for g'. Finding higher order derivs of g is equally easy: just differentiate more times in λ.

Bifurcation Theory is about what to do when this breaks down because $D_x F$ is not invertible.

For the elastica we have a pde (not an eqn in $\mathbb{R}^N$), but the "trivial solution" is especially simple, which helps us. To find a solution $\theta = \theta(\lambda, s)$, we can differentiate the pde in λ to get an eqn for

$$\dot{\theta} = \frac{\partial \theta}{\partial \lambda},$$

which should determine $\theta(\lambda, s)$ by integration in λ. [One can view this as a "continuation method" for solving the pde as λ varies.] Differentiating (*) gives

$$(**) \qquad A\dot{\theta}_{ss} + \lambda(\cos\theta)\dot{\theta} + \sin\theta = 0$$
$$\dot{\theta}(0) = 0, \quad \dot{\theta}_s(1) = 0.$$

If at a given λ, $\theta(\lambda, s) = 0$ for all s, then (**) gives

$$A\dot{\theta}_{ss} + \lambda\dot{\theta} = 0 \qquad 0 < s < 1$$
$$\dot{\theta}(0) = 0, \quad \dot{\theta}_s(1) = 0$$

If $\lambda <$ 1st eigenvalue of this "linearized eqn" then we conclude $\dot{\theta}(\lambda, s) = 0$ for $0 < s < 1$.

From now on let's take $A = 1$ for simplicity; then 1st eigenvalue is

$$\lambda_1 = \pi^2/4 \qquad \text{assoc to eigenfunction } \varphi(s) = \sin\left(\tfrac{\pi}{2}s\right)$$

and our calculation indicates that

$$\theta(\lambda, s) \equiv 0 \qquad \text{for } 0 < \lambda < \pi^2/4.$$

Of course, expt doesn't stop at $\lambda_0 = \pi^2/4$, and neither should we. A rigorous treatment (sketched later) uses "Lyapunov-Schmidt reduction". But to get a good understanding, let's first proceed more formally (along the lines of Antman §5.6 or Howell et al §4.9.3): let's look for

$$\lambda(\varepsilon) = \frac{\pi^2}{4} + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \cdots$$

$$\theta(\varepsilon, s) = 0 + \varepsilon\,\theta_1(s) + \varepsilon^2 \theta_2(s) + \cdots$$

and expand in powers of ε. Full eqn is

$$0 = (\varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots)'' + \left(\frac{\pi^2}{4} + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \cdots\right)\sin(\varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots$$

and using $\sin x \simeq x - \frac{1}{6}x^3 + \cdots$ we have

$$\sin(\varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots) = \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\left(\theta_3 - \frac{1}{6}\theta_1^3\right) + \cdots$$

so we get:

<u>at order ε</u> $\quad \theta_1'' + \frac{\pi^2}{4}\theta_1 = 0, \quad \theta_1(0) = \theta_1'(1) = 0$

$$\Rightarrow \theta_1(s) = g\,\varphi(s) \qquad \varphi = \sin\left(\frac{\pi}{2}s\right)$$

g = any constant

<u>at order ε^2</u> $\quad \theta_2'' + \frac{\pi^2}{4}\theta_2 = -\alpha_1\theta_1 = -\alpha_1 g\,\varphi(s),$

$$\theta_2(0) = \theta_2'(1) = 0$$

Soln exists iff RHS is $\perp \varphi$, ie if

$$\int_0^1 \alpha_1 g\,\varphi^2(s)\,ds = 0$$

So (assuming $g \neq 0$) we need $\alpha_1 = 0$, and θ_2 is again an arbitrary multiple of φ.

<u>at order ε^3</u> $\quad \theta_3'' + \frac{\pi^2}{4}\theta_3 = -\alpha_2\theta_1 - \cancel{\alpha_1\theta_2} + \frac{\pi^2}{4}\cdot\frac{1}{6}\theta_1^3$

0

Soln exists iff RHS $\perp \varphi$, ie when

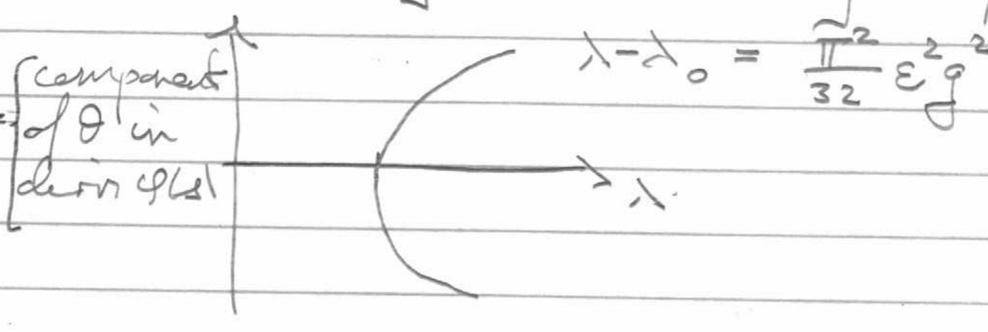
$$\int_0^1 -\alpha_2 g \varphi^2 + \frac{\pi^2}{24} g^3 \varphi^4 \, ds = 0$$

This simplifies to

$$\alpha_2 g = \frac{\pi^2}{32} g^3$$

since $\int_0^1 \sin^2(\frac{\pi}{2}s)\, ds = 1/2$, $\int_0^1 \sin^4(\frac{\pi}{2}s) = 3/8$.

We could continue, but there's no need: we've shown (formally) that bifurcation diagram is locally a parabola



since $\lambda - \lambda_0 \sim \alpha_2 \varepsilon^2$
and $g \neq 0 \Rightarrow \alpha_2 = \frac{\pi^2}{32} g^2$

(The bifurcation is called "supercritical" because the parabola opens to the right.)

Here's a sketch of how Liapunov-Schmidt reduction provides a more rigorous analysis (see Stakgold's SIAM Review article [vol 13, no 3, 1971, pp 289-332, esp sections 9+10] for more details]

Let's look for $\theta = g\varphi + \tilde{\theta}$, $\tilde{\theta} \perp \varphi$, where

$$\varphi = 1^{st} \text{ eigenfunction} = \sin(\tfrac{\pi}{2}s)$$

and orthogonality means $\int_0^1 \tilde{\theta}(s)\varphi(s)\,ds = 0$.

Write eqn $\theta'' + \lambda \sin\theta = 0$ as

$$\theta'' + \lambda_0\theta + (\lambda - \lambda_0)\theta + \lambda(\sin\theta - \theta) = 0,$$

ie

$$(1) \qquad \theta'' + \lambda_0\theta = -(\lambda-\lambda_0)\theta - \lambda(\sin\theta - \theta)$$

Consistency condn is

$$(2) \qquad (\lambda-\lambda_0)\theta + \lambda(\sin\theta - \theta) \perp \varphi$$

If this holds, there's a unique $\tilde{\theta} \perp \varphi$ solving (1). So we can view $\tilde{\theta} = \tilde{\theta}_{\lambda, g}$ as being defined (for λ near λ_0 and g near 0) by eqn (2) combined with

(3) $$\tilde{\theta}'' + \lambda_0 \tilde{\theta} = P_{\varphi^\perp}\left[-(\lambda-\lambda_0)\theta - \lambda(\sin\theta - \theta)\right]$$

$$\tilde{\theta} \perp \varphi, \quad \tilde{\theta}(0) = \tilde{\theta}'(1) = 0 .$$

Eqn (2) gives the reln between g + λ that describes the bifurcation diagram. One can show (using $\sin\theta - \theta \approx -\frac{1}{6}\theta^3$) that $\|\tilde{\theta}\|_{L^2} \leq C|g|^3$, so the leading order character of the bifurcation is

$$\int_0^1 (\lambda-\lambda_0)(g\varphi + \cancel{\tilde{\theta}})\varphi + \lambda_0\left(-\frac{1}{6}g^3\varphi^3\right)\varphi \, ds = 0$$

ie
$$(\lambda-\lambda_0)\, g \int_0^1 \varphi^2\, ds - \frac{1}{6}\lambda_0 g^3 \int_0^1 \varphi^4\, ds = 0$$

as we obtained earlier by expansion. (Essence of this approach: $\theta = g\varphi + \tilde{\theta}$ represents the nontrivial solutions as a graph over the 1D axis $g\varphi$.)

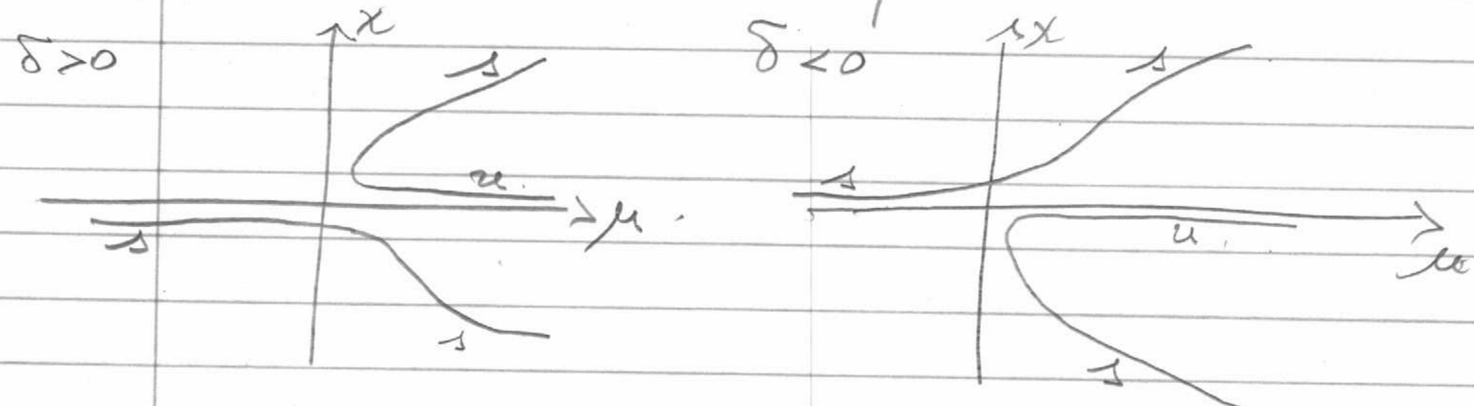
<u>Imperfection sensitivity</u>: an important aspect of bifurcation is that a small imperfection in the system will (typically) change the bifurcation diagram in an <u>essential</u> way. This is important, because

real systems are usually imperfect.

Simplest example: let's perturb $x^3=\mu x$ (where $x, \mu \in \mathbb{R}$; note that it describes the critical pts of $\frac{1}{4}x^4-\frac{\mu}{2}x^2$, which is convex for $\mu<0$ but not for $\mu>0$, so $\mu=0$ is the value where a bifurcation occurs); consider the perturbed problem

$$x^3-\mu x+\delta=0$$

(which describes critical pts of $\frac{1}{4}x^4-\frac{\mu}{2}x^2+\delta x$). Presence of $\delta\neq 0$ "breaks" the bifn diagram into two connected components.



(The labels "s" and "u" correspond to local min [stable] or non-local-min [unstable] critical pts of $\frac{1}{4}x^4-\frac{\mu}{2}x^2+\delta x$, with μ + δ held fixed. Since x is one-dimensional here, the unstable crit pt is a max.)

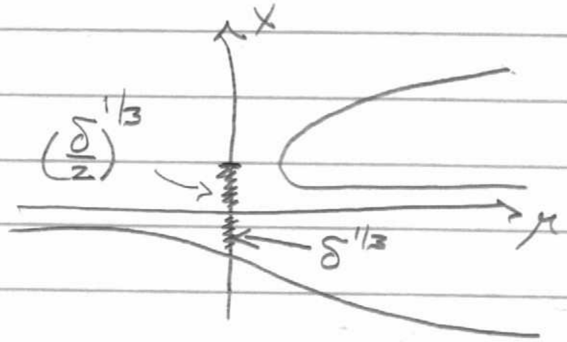
<u>Note</u>: effect of $\delta \neq 0$ involves fractional powers of δ (thus: influence of δ is not so small, even if δ is small); for example

$$\mu = 0 \implies x^3 = -\delta \implies x = (-\delta)^{1/3}$$

Also: vertical tangent in bifn diagram

$$\iff d\mu/dx = 0$$
$$\iff 3x^2 = \mu \iff x = (\mu/3)^{1/2}$$
$$\mu^{3/2} = \frac{3\sqrt{3}}{2}\delta$$

$\delta > 0$



A problem in HW 2 asks you to show that the buckling of an elastica with a small intrinsic curvature is essentially the same as this example. (Gravity is another physically natural imperfection; see Howell et al chap 4, prbm 4.17.)

<u>Digression</u>: In the late 70's people asked whether we can "classify all possible ways an imperfection can break the bifn

diagram". Answer was yes in many cases; see eg Golubitsky + Schaeffer, "A theory for imperfect bifurcation via singularity theory," CPAM 32 (1979) 21-98.

A start toward 3D (nonlinear) elasticity (if time permits). [Recommended reading:

- 1st 20 pages of book by Marsden + Hughes

- chapter 5 of Howell et al]

Big picture: we'll need eventually to discuss

a) strain (ie, description of stretching or shrinking assoc to a given deformation) &

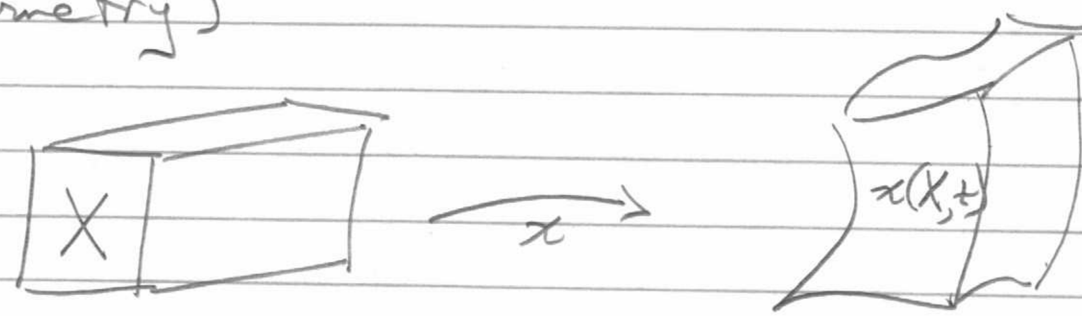
b) stress (ie description of forces by which one part of body acts on the rest)

c) balance laws (ultimately coming

from cons of linear + angular momentum)

d) constitutive laws (the reln b'twn stress + strain, which characterizes the material under consideration).

Start with strain (analysis of deformation; "kinematics"; closely tied to differential geometry)



$\Omega \subset \mathbb{R}^3$
reference ("Lagrangian") coordinates

deformed ("Eulerian") coordinates

$X = \{X_\alpha\}_{\alpha=1}^3$ $\qquad$ $x = \{x_i\}_{i=1}^3$

(conventions on labeling coords help us avoid confusion b'twn Lagrangian + Eulerian coords)

$$F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha} = \text{"deformation gradient"}$$

(linear approx of $X \to x(X)$)

Note: $X \to x(X)$ should be injective (no interpenetration of matter) so we expect $\det F > 0$.

<u>Polar decomposition</u>: if $\det F > 0$ then F can be written uniquely as

$$F = RU$$

where R is an orientation-preserving rotation + U is a pos def symmetric matrix; in fact

$$U = (F^T F)^{1/2}$$

Eigenvalues of U are "principal stretches" (eigenvectors are in the "principal directions")

Nonlinear strain is $U - I$. (Some authors would use $F^T F - I$ instead; not very different if U is close to I.)

Question 1: Which matrix-valued fns $F_{i\alpha}(X)$ are deformation gradients?

Answer is standard: in a simply connected domain, $F_{i\alpha} = \partial x_i / \partial X_\alpha$ iff each row is curl-free.

[Note: when studying dislocations in crystals, reqt that $F = \partial x / \partial X$ be curl-free is relaxed; presence of nonzero curl reflects presence of dislocations in the lattice.]

Question 2: Which matrix-valued fns $g_{\alpha\beta}(X)$ arise as $(F^T F)_{\alpha\beta} = \sum_{i=1}^{3} F_{\alpha i} F_{i\beta}$ for some $F_{i\alpha} = \partial x_i / \partial X_\alpha$?

Answer is more subtle, but still classical: $F^T F$ is a Riemannian metric – namely the metric of $\mathbb{R}^3$ (the space x), expressed in local coords X rather than x:

$$dx = F\,dX \Rightarrow |dx|^2 = |F\,dX|^2$$
$$= \langle F^T F\,dX, dX \rangle$$

In the language of differential

geometry: a pos def-symmetric-matrix-valued map $g_{\alpha\beta}(X)$ can be expressed as F^TF iff it is a "flat metric" (ie the standard metric on $\mathbb{R}^3$ in a possible weird coordinate system). Basic fact from diff'l geometry: $g_{\alpha\beta}$ is flat iff its "Riemann curvature tensor" vanishes. (This is a nonlinear pde system involving $g_{\alpha\beta}$ and its 1^{st} and 2^{nd} derivatives.) ✓

Question 3: Which deformations do no stretching or shrinking?

Answer: strain $= 0 \Rightarrow F^TF = I$
$\Rightarrow F = \partial x/\partial X$ is a rotn-valued fn of X.

Assuming a little smoothness, this implies $x(X)$ is a rigid motion. (This is another classical result from differential geometry.)

Question 4: Is there a "quantitative version" of the last result?

Answer is yes; for example, in a

cube, if $\text{dist}\left(\frac{\partial x}{\partial X}, SO(3)\right) \in L^2(Q)$ then $\exists$ rotn R st

$$\int_Q \left|\frac{\partial x}{\partial X} - R\right|^2 dx \le C \int_Q \text{dist}^2\left(\frac{\partial x}{\partial X}, SO(3)\right) d$$

[key pt: R is a constant rotation!]. This "nonlinear Korn inequality" is not so classical – it was proved by Friesecke, James, + Müller only in 2002.