# Mechanics – Lecture 2, 2/1/2022

Today's topic: some 1D problems involving resistance to bending. Also (if time permits) we'll begin discussing bifurcation in the context of Euler's elastica. (That will be continued in Lecture 3.)

Suggested readings: Antman's Chap 4 has a good discussion of beams + rods; his Chapter 5 provides an introduction to bifurcation. For something shorter (and more like these notes) see §4.9 of Howell-Kozyreff-Ockendon.

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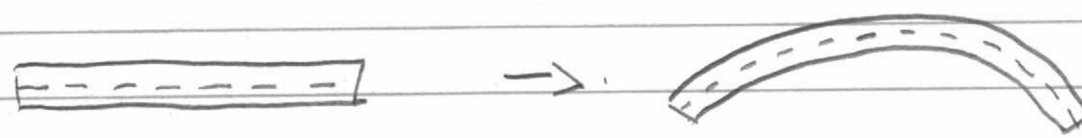
Big picture:

- In 2D, there is resistance to bending even for a sheet that's inextensible (eg paper, see the "xerox paper problem" discussed later in these notes, which you'll explore in HW2)

Essential mechanism: if a strip is mapped to an annulus with the midline mapping isometrically, then the lines parallel to the midline are stretched or shrunk due to the effects of curvature



- for a rod in 3D (or a ribbon, ie a rod with a rectangular cross-section whose width/height ratio is extreme) the picture is similar except that a rod can bend in <u>two</u> possible directions, and it can also <u>twist</u>.

To keep things simple, we'll focus on an <u>inextensible</u> <u>beam</u> (initially straight and uniform, eg a piece of paper or a diving board or a ruler), deformed by bending consistent with a 1D model.

We must discuss:

a) kinematics (description of deformation)

b) statics (forces + bending moments, and the assoc balance laws)

c) constitutive law (in this case: the relation between curvature and bending moment)

<u>Kinematics</u>: for a 1D inextensible rod that stays in the plane, this is easy: use $0<s<L$ as the reference variable, and $\vec{r}(s) \in \mathbb{R}^2$ as its deformed position. Since $|\vec{r}_s|=1$ (the hypothesis of inextensibility), we can take

$$\vec{r}_s = (\cos\theta(s), \sin\theta(s)).$$

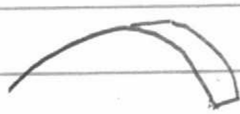
The rod's <u>curvature</u> is $\theta'(s)$; it will play a role similar to that of the "strain" $|\vec{r}_s|-1$ in our discussion of strings.

<u>Statics</u>: slicing the rod at $s=s_0$, each side acts on the other by

(i) a net force $\vec{n}(s)$ (as in a string)

(ii) a bending moment $\vec{m}(s)$

[Think of a curved piece of paper; if we ignore gravity its configuration will be part of a circle, as we'll see on HW2.]



Here $\vec{n}=0$ and $\vec{m}(s)$ is what maintains the bent shape ]

Defn of $\vec{m}$: the part of the beam at $s>s_0$ exerts torque $\vec{r}(s_0)\times\vec{n}(s_0)+\vec{m}(s_0)$ on the part at $s<s_0$. Note that for deformations + forces in the $x_1-x_2$ plane $\vec{m}=(0,0,M(s))$ is essentially scalar valued.

Balance of forces: $\vec{n}(s_1)-\vec{n}(s_0)+\int_{s_0}^{s_1}\vec{f}(s)\,ds=0$

where $\vec{f}$ is the force per unit reference length

Balance of torques:

$$\left[\vec{m}(s_1)+\vec{r}(s_1)\times\vec{n}(s_1)\right]-\left[\vec{m}(s_0)+\vec{r}(s_0)\times\vec{n}(s_0)\right]$$
$$+\int_{s_0}^{s_1}\vec{r}(s)\times\vec{f}(s)\,ds=0.$$

(assuming there's no additional physics introducing a source of torque per unit length)

These hold for all $s_0, s_1$ $\Rightarrow$

$$\vec{n}_s + \vec{f} = 0$$

$$\vec{m}_s + (\vec{r} \times \vec{n})_s + \vec{r} \times \vec{f} = 0$$

or equivalently

$$\boxed{\begin{aligned} &\vec{n}_s + f = 0 \\ &\vec{m}_s + \vec{r}_s \times \vec{n} = 0 \end{aligned}}$$

(Where does balance of torque come from? Well, dynamic version of balance of forces is essentially force = mass × acceleration, which comes from <u>conservation of linear momentum</u>. So it shouldn't be surprising that balance of torque comes from <u>conservation of angular momentum</u>. This will be clearer when we get to Hamiltonian mechanics; for now please take the balance laws as a starting point.)

<u>Notes</u>:

(1) If $f = 0$ then $\vec{n}$ is constant.

(2) The balance laws must be supplemented by bdry conditions.

For example: a 1D rod can be held in a bent position

either by applying forces at the endpts (horizontal, in the figure) or by applying bending moments at the endpts.

<u>Constitutive law</u>: since the rod is inextensible there is no constitutive law for $\vec{n}$ — instead, we get it by integration of $\vec{f}$ (+ using bdry conds).

The simplest law for $\vec{m}$ is the "physically linear" law

$$\vec{m} = (0,0,M), \quad M = A\theta'$$

where $A$ is constant. One can of course consider more nonlinear laws (taking $M$ to be a monotone function of $\theta'$, with $M(0)=0$). But the linear law is rich enough to be interesting; and for thin bodies it is quite appropriate since the stretching or shrinking produced by curvature is small in magnitude.

Specializing to $f = 0$, we get the model known as "Euler's elastica." (Euler discussed it in 1727, but actually J. Bernoulli considered it in 1694; Antman has lots on the history):

$$\vec{r}_s = (\cos\theta(s), \sin\theta(s))$$

$$\vec{n}_s = 0 \implies \vec{n} = \text{const} = -\Lambda(\cos\alpha, \sin\alpha)$$
for some $\Lambda, \alpha$

$$\vec{m}_s + \vec{r}_s \times \vec{n} = 0, \quad \vec{m} = (0, 0, M) \implies$$

$$(0, 0, M_s) = \Lambda(\cos\theta, \sin\theta, 0) \times (\cos\alpha, \sin\alpha, 0)$$

$$\implies M_s = \Lambda(\cos\theta\sin\alpha - \cos\alpha\sin\theta)$$

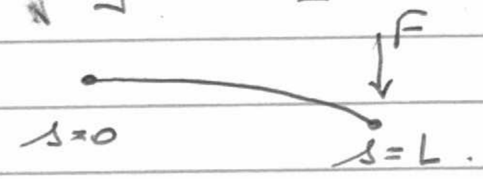
$$\implies M_s = M_s = \Lambda\sin(\alpha - \theta).$$

Constit law says $M_s = (A\theta'(s))'$. So ODE becomes

$$A\gamma''(s) + \Lambda\sin\gamma(s) = 0$$

where $\gamma(s) = \theta(s) - \alpha$. Note that $\Lambda$ and $\alpha$ are unknown (just as $\gamma(s)$ is unknown); they must be determined using the bdry conditions.

Example 1: deflection of a diving board (neglecting gravity)




Take $s=0$ to be clamped horizontally (so $\theta(0)=0$) and suppose a downward force $F$ is applied at RH edge $s=L$, which is otherwise free ($\theta'(L)=0$). Then $\vec{n}=(0,-F)$ and

$$(0,0,M_s) = (\cos\theta, \sin\theta, 0)\times(0,F,0)$$
$$= F\cos\theta$$

so

$$A\,\theta_{ss} = F\cos\theta \qquad 0<s<L$$

with $\theta(0)=0$, $\theta_s(L)=0$.

This has a variational principle: guessing a bit, we might expect to minimize

$$\underbrace{\int_0^L \tfrac{1}{2}A(\theta')^2\,ds}_{\text{elastic energy due to bending}} - \underbrace{(0,-F)\cdot\vec{r}(L)}_{\text{work done by load}}$$

Taking $\vec{r}(0)=(0,0)$, we have

$$\vec{r}(L) = \int_0^L \vec{r}_s \, ds = \int_0^L (\cos\theta(s), \sin\theta(s))\, ds$$

so the proposed var'l principle becomes

$$(*) \qquad \int_0^L \tfrac{1}{2} A \theta_s^2 + F\sin\theta \; ds$$

Its Euler-Lagrange eqn is indeed

$$-A\theta_{ss} + F\cos\theta = 0.$$

The functional $(*)$ must be minimized subject to $\theta(0)=0$ (the condition $\theta_s(L)=0$ is the "natural bdry cond" we get at $s=L$ when the value of $\theta$ is free there).

A more or less exact soln is possible:

$$A\theta_{ss}\theta_s = F\theta_s\cos\theta$$

$$\Rightarrow \quad \theta_s^2 = \frac{2F}{A}\sin\theta + \text{const}$$

$$= \frac{2F}{A}\left[\sin\theta + \mu\right]$$

where $\mu = -\sin\theta(L)$, using $\theta'(L)=0$. Note that we expect $d\theta/ds < 0$ so $\mu > 0$

A bit of calculus gives

$$\int_0^{-\theta(s)} \frac{d\varphi}{(\sin\mu - \sin\varphi)^{1/2}} = s\sqrt{\frac{2F}{A}}$$

The value of $\mu$ is thus determined by

$$\int_0^{\mu} \frac{d\varphi}{(\sin\mu - \sin\varphi)^{1/2}} = L\sqrt{2F/A}$$

(the LHS has no explicit formula but can easily be evaluated numerically).

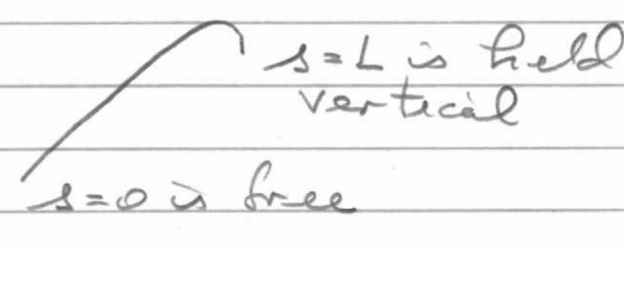
<u>Example 2</u>: The "xerox paper problem": describe the profile of a standard 8.5×11 sheet of paper, held at one edge so that the tangent there is vertical

Differences from the elastica:

- gravity matters
- bc will be different

Now $\vec{n}_s + f = 0$, $f = f_0(0,-1,0)$, with

conventions



The bc are:

- no force or moment at $s=0$
- specified angle $\theta = -\pi/2$ at $s=L$.

Evidently $\vec{n} = (a, b + f_0 s, 0)$ for some constants $a, b$. Since $\vec{n}(0)=0$ we have $a=b=0$,

$$\vec{n}(s) = (0, f_0 s, 0)$$

Now eqn for $\vec{m}$ and linear constitutive law give

$$A\theta'' + f_0 s \cos\theta(s) = 0 \qquad 0 < s < L.$$
$$\theta'(0) = 0, \quad \theta(L) = -\pi/2.$$

This too has a variational principle. We expect it to involve

$$\int_0^L \underbrace{\tfrac{1}{2} A \theta_s^2}_{\text{elastic energy}} + \underbrace{f_0 r_2(s)}_{\text{work done by load}} \, ds$$

Now, if $r_2(L) = 0$ then $-r_2(s) = \int_s^L \sin\theta(t)\, dt$

so $$-\int_0^L r_2(s)\, ds = \iint_{0<s<t<L} \sin\theta(t)\, dt\, ds.$$

$$= \int_0^L t \sin\theta(t)\, dt$$

since $\int_{0<s<t} ds = t$. The proposed var'l principle is thus

$$\int_0^L \tfrac{1}{2} A\theta_s^2 - f_0 s \cdot \sin\theta(s)\, ds$$

for which the EL eqn is indeed

$$A\theta'' + f_0 \cdot s \cos\theta(s) = 0$$

This time the minimization is subject to the bc $\theta(L) = -\pi/2$ (The condition $\theta'(0)=0$ is the "natural bdry cond" that emerges because $\theta(0)$ is unconstrained).

(On HW2 you'll be asked to estimate the value of the dimensionless parameter $A/f_0L^3$ for a sheet of xerox paper.)

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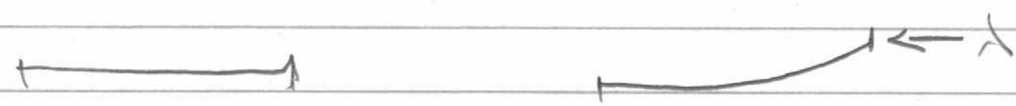
What to do with this that's interesting? <u>My choice</u>: let's consider an intuitive, physically natural example of <u>bifurcation</u>.

I'll introduce this topic in these notes, then continue next week.

[What to read on this topic: Antman's Chapter 5 is pretty good — it has versions of everything I'll do, and more. Howell-Kozyreff-Ockendon is concise but helpful, see their §4.9.3. Another good source is Ivar Stakgold's article "Branching of solutions of nonlinear eqns", SIAM Review 13 (1971) 289-332.]

<u>Goals</u>: consider the elastica with a compressive load $\lambda$. If $\lambda$ is large enough it will buckle. What is the critical load $\lambda_0$? How can we understand the buckled configurations?



$0 < s < 1$
reference

fixed
$\theta(0) = 0$
$\vec{r}(0) = 0$
at $s = 0$

Various bdry conds are possible at $s=1$; I'll focus on:

$\theta'(1) = 0$ — RHS is "pinned" to the loading device, so it rotates freely

(Thus: applied load at $s=1$ is $(-\lambda, 0)$ but applied bending moment is 0.)

Eqn is

$$A\theta'' + \lambda \sin\theta(s) = 0 \qquad 0<s<1$$
$$\theta(0)=0, \ \theta'(1)=0$$

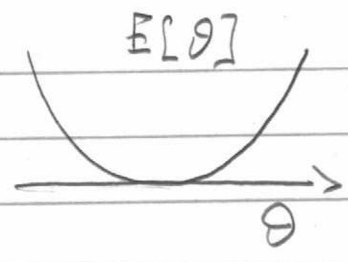
since $A\theta'' = -(\cos\theta, \sin\theta, 0) \times (-\lambda, 0, 0)$.
Var'l principle is

$$E[\theta] = \int_0^1 \tfrac{1}{2}A\theta_s^2 + \lambda\cos\theta \, ds$$

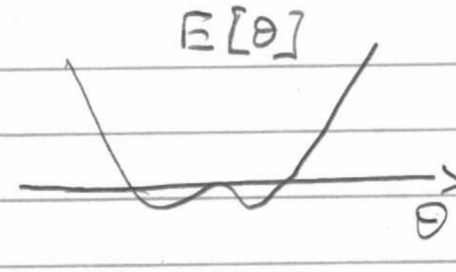
subject to $\theta(0)=0$ (by arg'ts similar to prior examples).

Clearly $\theta \equiv 0$ is a soln for any $\lambda$. When $\lambda$ is small we expect it to be stable; when $\lambda$ is larger we expect it to be unstable.

<u>Var'l viewpt</u>: for $\lambda > 0$, $\frac{1}{2}A\theta_s^2$ is convex but $\lambda\cos\theta$ is concave. For $\lambda$ near 0 we expect var'l pblm to have a unique crit pt at $\theta \equiv 0$ but if $\lambda$ is large enough we expect this critical pt to go unstable




$\lambda < \lambda_0$




$\lambda > \lambda_0$

In fact: approxn $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$ suggests

$$E[\theta] \simeq \int_0^1 \tfrac{1}{2} A \theta_s^2 - \tfrac{\lambda}{2}\theta^2 \, ds + \lambda .$$

Integral is positive ($E[\theta] \geq E[0] = \lambda$) iff

$$\int_0^1 \theta^2 \leq \frac{A}{\lambda} \int_0^1 \theta_s^2 \qquad \text{when } \theta(0) = 0$$

In fact

$$\min_{\theta(0)=0} \frac{\int_0^1 \theta_s^2}{\int_0^1 \theta^2} = \frac{\pi^2}{4}$$

(the extremal value is the first eigenvalue of $-\theta'' = \mu\theta$ with $\theta(0) = 0$ and $\theta'(1) = 0$; the first eigenfn is $\sin(\frac{\pi}{2} s)$, with eigenvalue $\pi^2/4$.)

So

$E[\theta]$ is strictly pos near $0 \Leftrightarrow \frac{\lambda}{A} \leq \frac{\pi^2}{4}$.

(To understand the local min that emerge for $\lambda > \frac{\pi^2 A}{4}$ we'll need to do more, of course).

<u>More mechanical viewpt</u>: a natural experiment is to gradually increase $\lambda$, starting from 0. This amounts mathematically to a "continuation method" for finding $\theta = \theta(s, \lambda)$. Implicit fn thm tells us that if we can find $\partial\theta/\partial\lambda$ then $\theta$ is locally uniquely determined. Now, diffn of pde wrt $\lambda$ gives (using $\dot{\theta}$ for $\partial\theta/\partial\lambda$)

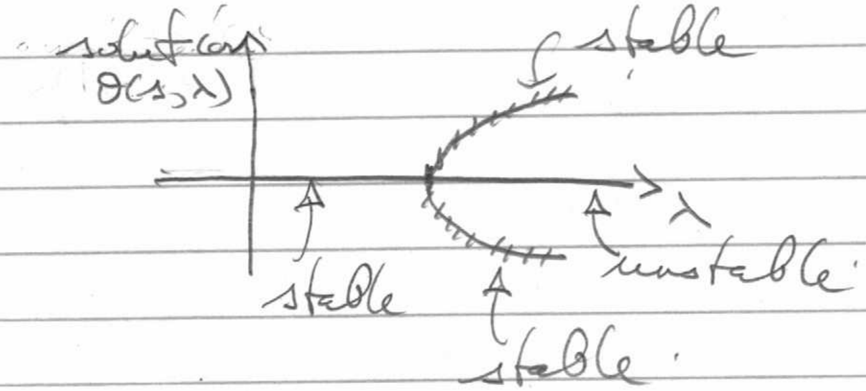
$$(**) \qquad A\dot{\theta}_{ss} + \lambda(\cos\theta)\,\dot{\theta} + \sin\lambda = 0$$
$$\dot{\theta}(0) = 0, \quad \dot{\theta}_s(1) = 0$$

If $\theta(s) \equiv 0$ then $(**)$ implies $\dot{\theta}(s) \equiv 0$ so long as $\lambda <$ 1st eigenvalue of

$$A\dot{\theta}_{ss} + \lambda\dot{\theta} = 0 \,;$$

as we found earlier, this requires $\lambda < \pi^2 A/4$.

But of course, expt doesn't stop at the critical value of $\lambda$. Instead, for larger $\lambda$ we'll see buckled configurations assoc to nonzero solns of the pde (local min of $E[\theta]$). In fact, the solns for $\lambda \approx \pi^2 EI/4$ can be visualized as




Here too, we need a new idea to understand the character of the solution & the geometry of the "bifurcation diagram".

To be continued in Lecture 3.