Mechanics - Lecture 1, 1/25/2022

(First of about 7 lectures on elasticity.)

The big picture: elasticity is concerned with

stress (forces)

strain (stretching + compression; more generally, description of deformation)

constitutive relations (what stress is produced by a given strain; this describes a material's elastic response)

balance laws (conservation of linear momentum + conservation of angular momentum ⇒ eqns of elastodynamics, or [as a special case] eqns of elastostatic equilibrium

elastic energy (special form of constitutive reln in 3D, but no loss of generality in 1D; leads to links with the calculus of variations)

<u>Examples</u> of things we might model this way:

- "elastic strings" – eg rubber band, or a steel cable [if bending resistance is not important]

- "elastic rods" – eg a plastic ruler or a flexible cylindrical rod [which resists bending + twisting]

- 2D elastic sheet, constrained to stay in the plane or else deformed in 3D – for example, a balloon made from rubber

- 3D elastic solid, for example a block of rubber

<u>Note</u>: most materials are inelastic at sufficiently large strain or stress, sometimes importantly so. Some types of inelastic behavior:

a) plasticity – large enough stress induces <u>permanent</u> deformation. (Easy to see in an aluminum paper clip.)

b) fracture – for example due to crack propagation. (Discussing this requires additional constitutive modelling: what determines the direction + speed of crack propagation?)

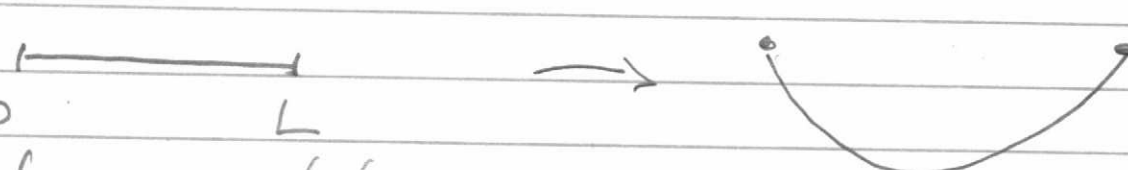
Dynamics introduces additional issues: we'll mostly stick to statics or conservative elastodynamics ("force = mass × acceleration") but sometimes dissipative dynamical laws are important, eg in

c) viscoelasticity – where the force depends on the deformation rate (silly putty is a great example; also automobile shock absorbers – viscous effects tend to dampen vibrations)

Some courses would begin with linear elasticity (advantage: linear pde's). Others with fully nonlinear elasticity of 3D solids (advantage: suitable for large deformations). We'll do both, but later.

My starting point: elastic strings, because the math is relatively elementary and it leads to good intuition, good exercises, and we can already see connections to the Calculus of Variations.
Good source: Antman's book Chaps 2+3 (of course he does much more than we'll have time for)

Getting started:



reference state of the string (parametrized by arclength, "unstretched")

deformed state of string at time t is $\vec{r}(s,t) \in \mathbb{R}^3$

$$\left|\frac{d\vec{r}}{ds}\right| = \nu(s,t) = \text{"stretch ratio"}$$

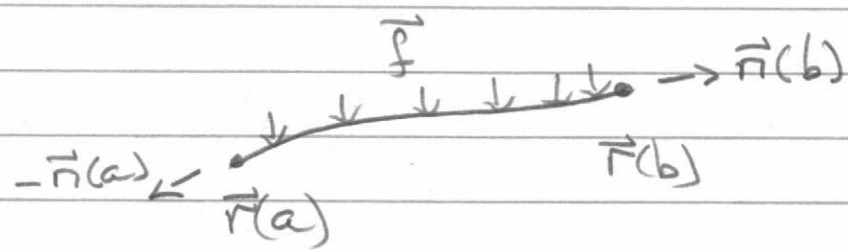
$\nu > 1 \iff$ tension
$\nu < 1 \iff$ compression

(Actually, you can't really compress a string — since it has no resistance to bending it will buckle instead. Thus any equilibrium with $\nu < 1$ will be unstable. Such equilibria are nevertheless interesting, eg for the design of arches — see HW 1.)

<u>Balance of forces</u>: let

$\vec{n}(s_0)$ = force exerted by the part of the string at $s > s_0$ on the part at $s < s_0$

$\int_{s_0}^{s_1} \vec{f}(s)\,ds$ = total "body force" on the part of the string assoc $s_0 < s < s_1$ (Thus: $f(s)$ is the force per unit <u>reference</u> length)



Conservation of momentum ("force = mass times acceleration") says

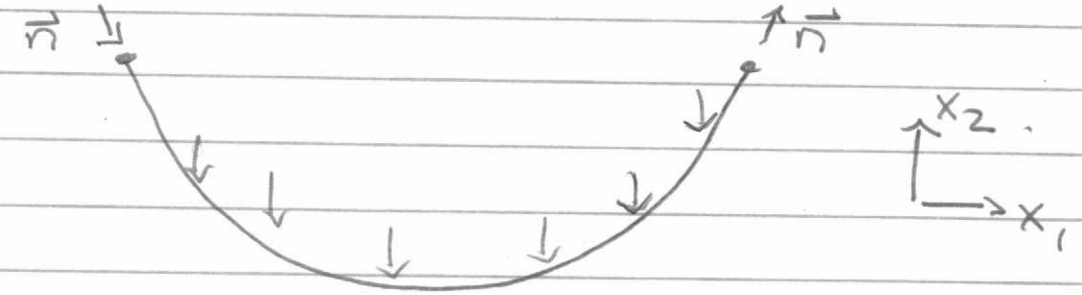
$$\vec{n}(b) - \vec{n}(a) + \int_a^b \vec{f}(s,t)\,ds = \int_a^b \rho(s)\,\vec{r}_{tt}(s,t)\,ds$$

where $\rho(s)$ = density of string (ie mass per unit reference length).

Assoc pde is

$$\vec{n}_s + \vec{f} = \rho\, \vec{r}_{tt} .$$

[Check correctness of sign: at equil $\vec{r}_{tt}=0$. For a string hanging down due to gravity we expect $f_2 < 0$ and $\partial_s n_2 > 0$ if direction 2 is vertical]



Constitutive law: to complete the picture (eg to get a pde that determines $\vec{r}(s,t)$) we need to specify the dependence of $\vec{n}$ on $\vec{r}_s$. (The string is "elastic" $\Longleftrightarrow$ $\vec{n}$ depends only on $\vec{r}_s$.) For an elastic string

① $\vec{n}$ is directed tangent to the string

② magnitude of $\vec{n}$ depends only on the "stretch" $|\vec{r}_s| = \nu$. (This is the "principle of frame indifference" in the special case of a string.)

So

$$\vec{n} = \hat{N}(\nu(s,t)) \frac{\vec{r}_s}{|\vec{r}_s|}$$

(assuming the string is uniform), where $\hat{N}(\cdot)$ is a function from $\mathbb{R}_+$ to $\mathbb{R}$. We usually expect

$\hat{N}(1) = 0$ (reference configuration is stress-free)

$\hat{N}(\nu) > 0$ for $\nu > 1$, with $\hat{N}(\nu) \to \infty$ as $\nu \to \infty$

$\hat{N}(\nu) < 0$ for $\nu < 1$, with $\hat{N}(\nu) \to -\infty$ as $\nu \to 0$.

[Remark: for a viscoelastic string the constitutive law would instead be rate-dependent: $\vec{n} = \hat{N}(\nu, \nu_t) \vec{r}_s / |\vec{r}_s|$.]

Eqn of equilibrium for a string is thus

$$\vec{n}_s + \vec{f} = 0, \quad \vec{n} = \hat{N}(|\vec{r}_s|) \vec{r}_s / |\vec{r}_s|$$

(a 2nd order nonlinear ode). It must of course be supplemented by boundary conditions. Typically these would specify, at each end, either $\vec{r}$ or $\vec{n}$ (but not both). (Other types of bdry conds are possible, eg one end of the string could be attached to a surface but slide freely along it.)

When is there an associated variational principle? In other words, when are equilibria critical pts of some functional? Var'l prins are important because

a) they often exist, and can help us analyze problems (as we'll see shortly) as well as to solve them numerically

b) getting bdry conds right is often easier using the var'l principle

c) they shed light on elastic stability (to be explained shortly).

For an elastic string it's natural to seek a var'l principle involving

$$F[\vec{r}] = \int_0^L W(|\vec{r}_s|)\,ds + \int_0^L V(\vec{r}(s))\,ds.$$

(Why $W(|\vec{r}_s|)$ not $W(\vec{r}_s)$? Because rotating the string in space doesn't change its mechanical state; this is the "principle of frame indifference" at the variational level.)

Let's find the Euler-Lagrange eqn for F, assuming the bdry cond is $\vec{r}(0)=\vec{a}$, $\vec{r}(L)=\vec{b}$.

Notation: $\delta F=0$ means $\frac{d}{dt}\Big|_{t=0} F[\vec{r}+t\vec{g}]=0$ for every g st $\vec{g}(0)=0$ and $\vec{g}(L)=0$.

Differentiate under integral to get assoc pde (we assume $|\vec{r}_s|$ stays away from 0 and ∞, so this is justified):

$$\frac{d}{dt}\Big|_{t=0} F(\vec{r}+t\vec{g}) = \int_0^L \langle W'(|\vec{r}_s|)\frac{\vec{r}_s}{|\vec{r}_s|}, \vec{g}_s\rangle\, ds$$
$$+\int_0^L \langle \nabla V(\vec{r}(s)), \vec{g}(s)\rangle\, ds.$$

But (using that $g=0$ at endpts) 1^{st} term is

$$-\int_0^L \langle \left(W'(|\vec{r}_s|)\frac{\vec{r}_s}{|\vec{r}_s|}\right)_s, \vec{g}\rangle\, ds$$

Now, if a function $\vec{\varphi}(s)$ satisfies

$$\int_0^L \langle \vec{\varphi}(s), \vec{g}(s) \rangle \, ds = 0 \quad \text{whenever } \vec{g}(0) = \vec{g}(L) = 0$$

Then we must have $\vec{\varphi}(s) \equiv 0$ (Exercise!).
So

$$\delta F = 0 \iff \frac{d}{ds}\left(W'(|\vec{r}_s|) \frac{\vec{r}_s}{|\vec{r}_s|} \right) - \nabla V(\vec{r}(s)) = 0.$$

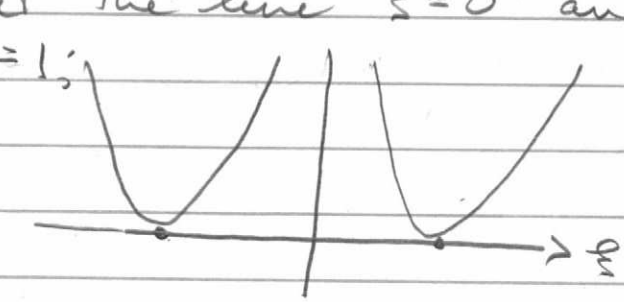
Conclusion: equil of elastic string has a var'l principle provided the force is "conservative", ie

$$\vec{f}(s) = -\nabla V(\vec{r}(s)) \quad \text{for some scalar-valued } V : \mathbb{R}^3 \to \mathbb{R},$$

(Note: gravity is conservative!) The assoc elastic energy W satisfies

$$W'(\nu) = \hat{N}(\nu), \quad \text{ie } W(\nu) = \int_1^\nu \hat{N}(\tau)\, d\tau.$$

Let's look at the character of the elastic energy $\int W(|\vec{r}_s|)\, ds$. Its integrand is <u>nonconvex</u>; in fact the graph of $\xi \to W(|\xi|)$ is symmetric about the line $\xi = 0$ and smallest at $|\xi| = 1$;
(the figure shows a slice of it)



If $\hat{N}'(\nu) > 0$ for $\nu \geq 1$ then there's a <u>convex</u> problem <u>below</u> this one, agreeing with it when $|\vec{r}_s| \geq 1$, namely

$$\text{"convexification of } W(|\vec{r}_s|)\text{"} = \begin{cases} W(|\vec{r}_s|) & \text{if } |\vec{r}_s| \geq 1 \\ 0 & \text{if } |\vec{r}_s| \leq 1 \end{cases}$$

Replacing $\int_0^L W(|\vec{r}_s|)\,ds$ by its convexification doesn't change the min value of the functional, but it eliminates lots of critical pts (in some settings); if minimizer found using convexified functional has $|\vec{r}_s| > 1$ then we have solved the original problem as well.

A little more about nonconvexity and why it might matter: consider the rather analogous 1D var'l problem

$$\int_0^1 (u_x^2 - 1)^2\,dx \qquad \text{where } u: (0,1) \to \mathbb{R}.$$

Depending on details, minimizing this plus $\int_0^1 V(u)\,dx$ might not achieve its min; for example

$$\min \int_0^1 (u_x^2-1)^2 + u^2\,dx$$

has min value 0 but it isn't attained.

Returning to our F: if a min-energy state exists then it must not use compressive states (since introduction of buckling or "wiggles" would relieve the compression, thereby decreasing the elastic energy while barely changing the term involving V. So: for elastic string, replacing W by its convexification suffices for finding min energy states of the original problem as well.

Now let's explain why a local min of F gives a stable equilibrium. Consider, as an analogy, the nonlinear wave eqn

$$(*) \qquad u_{tt} = \partial_x \left[W'(u_x) \right] - V'(u)$$

where $u = u(x,t)$ is real-valued, defined on $0 < x < L$, with $u = 0$ at $x = 0$ and $x = L$. Equilibria are critical pts of

$$F(u) = \int_0^L W(u_x) + \int_0^L V(u)\, dx$$

(by the same argument we used earlier for the elastic string) and for the dynamical eqn we have

$$\frac{d}{dt}\int_0^L \underbrace{\tfrac{1}{2}u_t^2}_{\substack{\text{"kinetic}\\ \text{energy"}}} + \underbrace{W(u_x) + V(u)}_{\substack{\text{"potential}\\ \text{energy"}}}\, dx$$

$$= \int_0^L u_t u_{tt} + W'(u_x)u_{tx} + V'(u)u_t \, dx$$

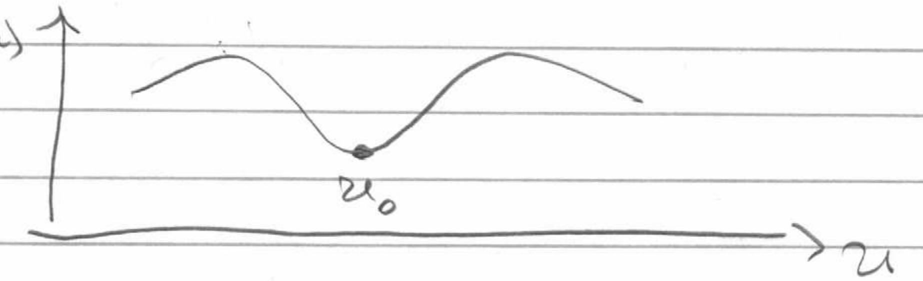
$$= \int_0^L \left\{ u_{tt} - [W'(u_x)]_x + V'(u) \right\} u_t \, dx$$

(using the bc to know there is no bdry term)

$$= 0 \qquad \text{by } (*)$$

Briefly: eqn (*) has a Hamiltonian structure and so kinetic + potential energy is conserved.

Now suppose $F[u] = \int_0^L W(u_x) + V(u)\, dx$ has a local min u_0 that's a <u>strict energy well</u>



in the sense that as one leaves u_0, the energy must strictly increase. Then solving (*) with initial cond u_0 (or even close to u_0) and initial $\int |u_t|^2 dx$ small.

$\Rightarrow$ $F(u)$ must remain $F(u_0)$ for all time (just from our hypoth on the strict energy well, together with positivity of $\int u_t^2$ and conservation of kinetic plus potential energy)

$\Rightarrow$ u must stay close to u_0

The same arguments apply to the elastic string (at least formally), using

$$\int_0^L \frac{1}{2}\rho |\vec{r}_t|^2 + W(|\vec{r}_s|) + V(\vec{r}) \; ds$$

for the total energy.

[Making this honest for the string has some subtlety, however, since the notion of a "local min" requires a choice of topology. For nonconvex problems, the choice of topology can matter! For example, in the var'l problem

$$\Phi[u] = \int_0^1 (u_x^2 - 1)^2 dx \qquad \text{where } u:[0,1]\to\mathbb{R},\; u(0)=0,\; u(1)=\alpha$$

we can ask: is $u_*(x) = \alpha x$ a local min of Φ? If α is chosen in $[-1,1]$ st $W''(\alpha) > 0$ then the answer is: $u_*(x)$ is a local min in the C^1 topology, but not in the L^∞ topology.

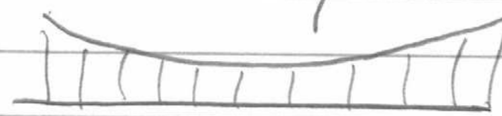
Worse: if, in our discussion of (*), $W''(u_x)$ is not positive, then well-posedness of the evolution is not clear.

But: we have argued that for the string, an energy-minimizer should not involve compression. So our stability discussion should be applied using the "convexified" elastic energy.]

What types of bvp might we want to solve for an elastic string? HW 1 explores a few of them:

① "catenary pbm" – a string suspended from its endpts, pulled down by gravity: what curve does it achieve? (Analogous issues arise also in the design of arch-shaped structures.)

(2) "suspension bridge pbm" — like (1) but now load is per unit horizontal distance (like a suspension bridge)



(3) "pressurized ring" — a simple closed curve under uniform pressure (pos or neg) — does it form a circle?