Mechanics - Lecture 9 - 3/29/2022

Reminder where we are: the two alternative perspectives on classical mechanics are

(1) the Lagrangian perspective: trajectories solve the EL eq'ns of a var'l pblm

$$\int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

where $L : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is convex in the 2nd variable (with faster-than-linear growth as $|\dot{q}| \to \infty$). In other words, they solve

$$\frac{\partial L}{\partial q_i}(q, \dot{q}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) = 0$$

where each term is evaluated at $q(t)$, $\dot{q}(t) = \frac{d}{dt} q(t)$.

(2) the Hamiltonian perspective: trajectories solve an ODE system of form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

where $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is convex in p (again, a growth cond is needed ~~two~~ as $|p| \to \infty$).

Equivalence of these viewpts is far from obvious in general, but for interacting particles solving

$$m_i \ddot{x}_i = -\frac{\partial}{\partial x_i} U(x_1, \cdots, x_N)$$

it is elementary:

$$H = \sum_i \frac{1}{2} m_i^{-1} |p_i|^2 + U(q_1, \cdots, q_N)$$

$$p_i = m_i \dot{x}_i \;, \quad q_i = x_i$$

$$L = \sum_i \frac{1}{2} m_i |\dot{q}_i|^2 - U(q_1, \cdots, q_N)$$

$$q_i = x_i \,, \; \dot{q}_i = \dot{x}_i$$

We'll discuss the equivalence in greater generality pretty soon, but first let's explore the implications + uses of these viewpoints.

We started last wk with the Lagrangian approach. A key takeaway (implicit but not explicit in that discussion) is that when there are no forces on a particle except the constraint to stay on a hypersurface, the

(Lagrangian) trajectories are constant-speed geodesics. Another takeaway was how easy it is to change coordinates.

A different use of Lagrangian perspective is to see how <u>invariances of the Lagrangian lead to conservation laws</u>.

Before starting that, let's review the entirely elementary facts that

a) For any Hamiltonian system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p} = \frac{-\partial H}{\partial q_i}, \quad \frac{d}{dt} H(q(t), p(t)) = 0$$

[the pf is just chain rule]

b) For masses interacting by a pair potential

$$m_i \ddot{x}_i = -\frac{\partial}{\partial x_i} \sum_{j \neq k}' V(|x_j - x_k|)$$

linear momentum is conserved (assuming m_i = const in time):

$$\frac{d}{dt} \sum_i m_i \dot{x}_i = \sum_i m_i \ddot{x}_i$$

$$= -\sum_{i \neq k}' V'(|x_i - x_k|) \frac{x_i - x_k}{|x_i - x_k|}$$

$$+ \sum_{i \neq j}' V'(|x_j - x_i|) \frac{x_j - x_i}{|x_j - x_i|}$$

$$= 0$$

Note: in this setting Newton's law says.

$$m_i \ddot{x}_i = \sum_{j \neq i}' F_{ij} \quad \text{where } F_{ij} \parallel x_i - x_j$$
$$\text{and } F_{ji} = -F_{ij}$$

and preceding argt amounted to
$\sum_i' \sum_{j \neq i}' F_{ij} = 0$ (since $F_{ij} = F_{ji}$).

c) In the setting of (b) restricted to masses moving in $\mathbb{R}^3$, angular momentum is also conserved. [Why just $\mathbb{R}^3$? Well, angular momentum is $\sum' x_i \wedge m_i \dot{x}_i$. We need the vector crossproduct to define it!]

$$\frac{d}{dt} \sum_i' x_i \wedge m_i \dot{x}_i = \sum_i' x_i \wedge m_i \ddot{x}_i$$

(since $\dot{x}_i \wedge \dot{x}_i = 0$)

$$= \sum_i' \sum_{j \neq i} x_i \wedge F_{ij}$$

Now, if $i \neq j$, $F_{ij} = c_{ij}(x_j - x_i)$ with $c_{ij} = c_{ji}$, so

$$x_i \wedge F_{ij} + x_j \wedge F_{ji} = c_{ij}\left[x_i \wedge (x_j - x_i) + x_j \wedge (x_i - x_j)\right]$$

$$= c_{ij}\left[x_i \wedge x_j - x_j \wedge X_i \right]$$

$$= 0$$

so $\sum_i \sum_{j \neq i} X_i \wedge F_{ij} = 0.$

Presented this way, these laws look like mere accidents. Emmy Noether was the first to understand that they are special cases of a much more general principle, by which any one-parameter family of invariances of the Lagrangian leads to a conservation law.

Peter Olver has a nice talk about this result on his website
www-users.cse.umn.edu/nolver/t_/noetherth.pdf
(a 2015 talk). Rather than present a full treatment, I'll try to give the flavor of this result through some examples.

Example 1: Reduction of a 2D particle in a central force field to an ODE for $r(t)$ only: suppose a Newtonian particle in $\mathbb{R}^2$ experiences force assoc to a potential $U(r)$ with $r = |x|$. Using polar coords, and rembering that

$$\dot{x} = \dot{r}\, e_r + \dot{\theta}\, r\, e_\theta$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - U(r)$$

and EL eqn for $\int L(r, \theta, \dot{r}, \dot{\theta})\, dt$ is

$$\frac{\partial L}{\partial r} = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}}, \qquad \underbrace{\frac{\partial L}{\partial \theta}}_{0} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}$$

$$\Rightarrow \quad -U'(r) + m r \dot{\theta}^2 = \frac{d}{dt}(m \dot{r})$$

$$0 = \frac{d}{dt}(m r^2 \dot{\theta})$$

Assuming $m = \text{const}$, this gives

$$m r^2 \dot{\theta} = c_0 \quad \Rightarrow \quad \dot{\theta} = \frac{c_0}{m r^2}$$

$$\Rightarrow \quad m \ddot{r} = -U'(r) + \frac{c_1}{r^3}$$

with $c_1 = \frac{c_0^2}{m}$. In particular, c_1 is constant (determined by m and the initial values of $r + \dot{\theta}$).

Essential mechanism of this calculation: the independence of L wrt θ (that is, the invariance of L wrt rotation about 0) led to the conservation law $r^2\dot{\theta} = \text{const}$

Example 2: If $L(q,\dot{q})$ has no explicit dependence on time then trajectories "conserve energy" in the sense that

$$E = \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L$$

is constant in time. [Note: we'll see later that if $L = T - U$ with $T = \frac{1}{2}\langle a(q)\dot{q}, \dot{q}\rangle$ for some symmetric-matrix-valued $a(q)$, then the assoc Hamiltonian system uses $p_i = \sum_j a_{ij}\dot{q}_j$ and $H = \frac{1}{2}\langle a^{-1}p, p\rangle + U = T+U$. In this setting $\langle \dot{q}, \nabla_{\dot{q}} L\rangle = 2T$ so $E = 2T - L = 2T - (T-U) = T+U$ is the Hamiltonian, and this law is the familiar fact that $H = \text{const}$ along trajectories].

Pf: EL eqn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$ gives (since L has no explicit time dependence)

$$\frac{d}{dt} L = \sum{}' \frac{\partial L}{\partial q_i} \dot{q}_i + \sum{}' \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

$$= \sum{}' \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i + \sum{}' \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

$$= \frac{d}{dt}\left(\sum{}' \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

<u>Example 3</u>: Cons of linear momentum comes similarly from translation invariance, ie hypoth that for any $\vec{a} \in \mathbb{R}^3$,

$$\frac{d}{dt} L(q_1 + t\vec{a}, \cdots, q_N + t\vec{a}\,;\, \dot{q}_1, \cdots, \dot{q}_N) = 0.$$

In fact this gives

$$\sum_{i=1}^{N}{}' \frac{\partial L}{\partial q_i} = 0 \quad \text{as a vector in } \mathbb{R}^3$$

Combining this with the EL eqn $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ gives

$$\sum_{i=1}^{N}{}' \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

ie

$$\sum_{i=1}^{N}{}' \frac{\partial L}{\partial \dot{q}_i} \quad \text{is constant along trajectories.}$$

Example 4: Cons of angular momentum comes from rotational invariance of the Lagrangian, ie from

$$L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N)$$

$$= L(Rq_1, \dots, Rq_N; R\dot{q}_1, \dots, R\dot{q}_N)$$

where q_i = posn of i^{th} particle in $\mathbb{R}^3$, and R = any rotn of $\mathbb{R}^3$. Arg't is not unlike that of Example 3; see eq 29 of Landau + Lifshitz for details. Here I'll present just a special case, whose proof captures the main idea:

(*) for particles in $\mathbb{R}^3$ moving by

$$m_i \ddot{x}_i = -\frac{\partial}{\partial x_i} U(x_1, \dots, x_N),$$

if U is invariant under rotn abt $\vec{e}$ axis then the $\vec{e}$ component of angular momentum is conserved.

Proof of (*): recall that $L = \sum_i \frac{1}{2} m_i |\dot{q}_i|^2 - U$, and

$$\frac{d}{dt}\sum_i{}' [x_i \wedge (m_i \dot{x}_i)] \cdot \vec{e}$$

$$= \frac{d}{dt}\sum_i{}' \left[q_i \wedge \frac{\partial L}{\partial \dot{q}_i}\right] \cdot \vec{e}$$

$$= \sum_i{}' \left[\cancel{\dot{q}_i \wedge \frac{\partial L}{\partial \dot{q}_i}} + q_i \wedge \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \cdot \vec{e}$$

0 since

$$\frac{\partial L}{\partial \dot{q}_i} = m_i \dot{q}_i$$

$$= \sum{}' \left[q_i \wedge \frac{\partial L}{\partial q_i}\right] \cdot \vec{e}$$

since

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Now we use the fact that if $\varphi_t(x) =$ rotn of x by angle t about $\vec{e}$ axis, $\frac{d}{dt}\varphi_t(x) = \vec{e} \wedge \vec{x}$. So our hypothesis on L says $\sum_i{}' \frac{\partial L}{\partial \dot{q}_i} \cdot (\vec{e} \wedge q_i) = 0$

Using the vector identity $a \cdot (b \wedge c) = -b \cdot (a \wedge c)$ [they're both $\det(a, b, c)$] we get

$$\left(\sum_i{}' q_i \wedge \frac{\partial L}{\partial q_i}\right) \cdot \vec{e} = 0.$$

So $(\sum{}' x_i \wedge m_i \dot{x}_i) \cdot \vec{e}$ constant along trajectories.

The Hamiltonian viewpoint also has important consequences (which are non-obvious even for the simplest examples, such as particles in $\mathbb{R}^3$ interacting by conservative forces).

One is <u>Liouville's Thm</u>: the flow

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{-\partial H}{\partial q_i}$$

is <u>volume-preserving</u> in the sense that it conserves the measure $dq_1 \cdots dq_N \, dp_1 \cdots dp_N$.

<u>Proof</u>: For any flow in Eucl space (in this case $\mathbb{R}^{6N}$ if the particles are in $\mathbb{R}^3$, but to start we can consider any flow $x \to \varphi_t(x)$ defined for $x \in \mathbb{R}^m$) we can consider its "infinitesimal generator"

$$\varphi_t(\vec{x}) = \vec{x} + t\,\vec{f}(\vec{x}) + O(t^2)$$

where $\vec{f}(\vec{x}) = \frac{d}{dt}\Big|_{t=0} \varphi_t(\vec{x})$ is a vector field.

The flow is volume-preserving iff $\vec{f}$ is divergence-free, since for any set D

$$\frac{d}{dt}\Big|_{t=0} (\text{vol of image of } D) = \int_D \frac{d}{dt}\Big|_{t=0} \det(I + t\nabla f)$$

$$= \int_D \operatorname{div} \vec{f} \;.$$

Applying this to the Hamiltonian flow using $\vec{x} = (\vec{q}, \vec{p})$ and $\vec{f} = \left(\frac{\partial H}{\partial p}, \frac{-\partial H}{\partial q}\right)$, we get

$$\operatorname{div} f = \sum \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i}\right) = 0.$$

Liouville's thm has some surprising consequences, eg

<u>Poincaré's recurrence thm</u>: if g is a vol preserving map (eg the time-one map of a Hamiltonian flow) and $g(D) = D$ for some set D of finite volume, then g is "recurrent" in the sense that

> for any set B of pos measure (eg a tiny ball), $\exists\, x_0 \in B$ st $g^n(x_0)$ is again in B for some $n < \infty$.

[Instructive examples: rotn of S^1 by a rational or irrational angle].

Proof of recurrence: clearly $B, g(B), g^2(B)$, etc cannot all be disjt (since they are subsets of D and D has finite volume).
So $\exists x_1$ st $x_1 \in g^k(B) \cap g^l(B)$ for some $l < k$.
Then $x_0 = g^{-k}(x_1)$ satisfies $x_0 \in B \cap g^{l-k}(B)$,
ie

$$x_0 \in B \quad \text{and} \quad g^{k-l}(x_0) \in B$$

as claimed.

Typical mechanical consequence: consider motion of a ball in an asymmetric bowl under gravity.

Region of phase space st kinetic + potential energy $\leq$ const is invariant & has finite volume in the assoc 4-dim'l phase space.
Take $B = \{$initial velocity near 0, initial posn near $x_*\}$ for any x_*. Then there's an x_0 near x_* and an initial velocity near 0 and an $n \in \mathbb{Z}_+$ st ball starting this way returns to be nearly at rest + nearly at x_0 after n iterations.

<u>One more consequence</u> of Hamiltonian framework is a scheme for "dimension reduction": if H is independent, say, of q_1 then problem reduces to Hamilton's eqns in $(q_2, \dots, q_N, p_2, \dots, p_N)$. In fact,

$$\frac{dp_1}{dt} = \frac{-\partial H}{\partial q_1} = 0 \quad \text{by hypoth,}$$

so p_1 = const along the trajectory. (The const would be determined by the initial data.) We can then solve

$$\dot{p}_j = \frac{-\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \quad (j \geq 2)$$

(substituting the constant value of p_1 to evaluate the RHS). Finally, $q_1(t)$ can be found by integrating

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}$$

along the resulting path.

This argument can be repeated. So if H depends on just one "spatial" variable then the evolution can be reduced to phase plane analysis. (Our Lagrangian

Example 1 was of this form; indeed, this dimension reduction idea amounts to a Hamiltonian perspective on that type of example. We'll show later that

$$H(q,p) = \sup_{\dot{q}} \langle p, \dot{q} \rangle - L(q,\dot{q})$$

and

$$L(q,\dot{q}) = \sup_{p} \langle \dot{q}, p \rangle - H(q,p)$$

so H is indep of $q_j \iff L$ is indep of q_j.)