Mechanics - Lecture 8 - 3/22/2022

New topic: 1st of several lectures on "classical mechanics," emphasizing links to calculus of variations and pde

Major goals include:

- some basic examples - to gain intuition + to see scope of theory + to see links with physics + pde
- Hamiltonian approach considers a class of ODE's; Lagrangian approach considers a class of var'l problems. Their equivalence is not obvious.
- Lagrangian approach is very close to optimal control; there the "value function" solves a Hamilton-Jacobi equation and the Hamiltonian trajectories are its characteristics.
- For physical systems, some things are easier to see using the Lagrangian viewpt (eg writing the eqns in a new coord system) but others are easier to see using the Hamiltonian one

(eg Liouville's thm)

My syllabus has lots of books. We won't follow any of them linearly. However

- Bühler's little book (chapter 1) and Mark Tuckerman's stat mech book (chapter 1) are perhaps the most useful sources, since their goals + viewpoints are a lot like mine

- initial chapters of Landau/Lifshitz, Arnold, + José/Saletan have <u>lots</u> more examples than we'll have time for (but maybe also more than you have time for).

The most basic (but still useful!) examples involve motion of interacting particles in $\mathbb{R}^n$ (eg $\mathbb{R}^3$):

(*) $\quad m_i \ddot{x}_i = f_i$

m_i = mass of i^{th} particle
$\ddot{x}_i$ = accel " "
f_i = force on i^{th} particle

(here $x_i(t) \in \mathbb{R}^n$). We assume the force is <u>conservative</u>

$$(**) \qquad f_i = -\frac{\partial U}{\partial x_i}$$ where $U(x_1, \ldots, x_N)$ is fn of positions of the particles

Key feature of (*), (**):

$$H = \underbrace{\tfrac{1}{2}\sum' m_i |\dot{x}_i|^2}_{\text{kinetic energy}} + \underbrace{U(x)}_{\text{potential energy}}$$

is conserved, since

$$\frac{dH}{dt} = \sum' m_i \langle \dot{x}_i, \ddot{x}_i \rangle + \sum' \langle \frac{\partial U}{\partial x_i}, \dot{x}_i \rangle$$

$$= 0 \quad \text{by } (*) \text{ \& } (**)$$

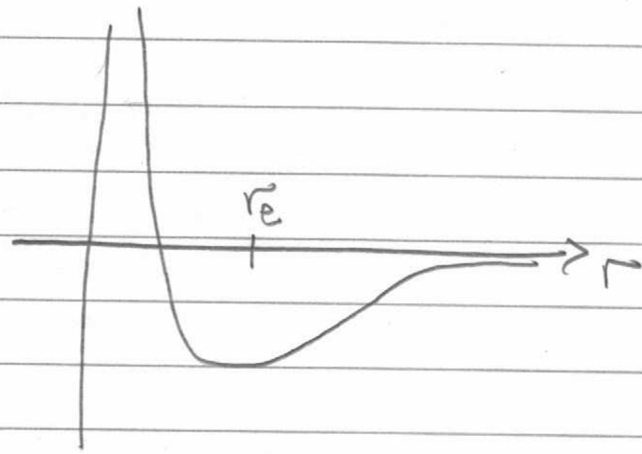
<u>Example 1</u>: particles interacting by pairwise attraction/repulsion

$$m\ddot{x}_i = -\sum'_{j \neq i} \frac{\partial}{\partial x_i} V(|x_i - x_j|)$$

Very widely used (eg for modeling fluids): the Lennard-Jones potential

$$V(r) = c\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

where σ, c are constants. Graph looks like



where r_e solves $\left(\frac{\sigma}{r_e}\right)^6 = \frac{1}{2}$. Particles repel when $|x_i - x_j| < r_e$ & attract when $|x_i - x_j| > r_e$, but attraction is negligible as $|x_i - x_j|/r_e \to \infty$; repulsion is strong as $|x_i - x_j|/r_e \to 0$

Example 2: wave eqn, discretized in space (but not time)

$$\ddot{w}_i = \frac{w_{i+1} + w_{i-1} - 2w_i}{(\Delta x)^2}$$

(If we're discretizing wave eqn on $0 < x < 1$ with $w = 0$ at $x = 0$ and $x = 1$, for example, then $w_1, \dots, w_{N-1}$ are nodal values at $x_j = j/N$ and $\Delta x = 1/N$.)

Here the potential is $U = \frac{1}{2} \sum' \left(\frac{W_i - W_{i-1}}{\Delta x} \right)^2$

Note: we can just as easily consider a nonlinear wave eqn. For example

$$\ddot{W}_i = \frac{W_{i+1} + W_{i-1} - 2W_i}{(\Delta x)^2} + W_i^p$$

is assoc to potential

$$U = \frac{1}{2} \sum' \left(\frac{W_i - W_{i-1}}{\Delta x} \right)^2 - \frac{1}{p+1} \sum' W_i^{p+1}$$

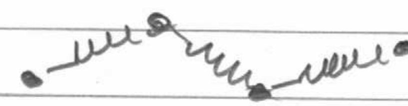
This eqn discretizes $W_{tt} = \Delta W + W^p$, of course. It is formally Hamiltonian with potential $\int \frac{1}{2} |\nabla W|^2 - \frac{1}{p+1} W^{p+1} \, dx$ and kinetic energy $\int \frac{1}{2} W_t^2 \, dx$.

Example 3: a system of masses + springs gives a Hamiltonian system much like example 1:

$$m_i \ddot{x}_i = - \sum'_{\substack{j \text{ attached} \\ \text{to } i \text{ by} \\ \text{spring}}} \frac{\partial}{\partial x_i} V(|x_i - x_j|).$$

A "linear spring" would have $V(r) = \frac{1}{2} K |r - r_e|^2$.

Often used as a simple model of a polymer



A lattice of springs (including diagonal springs) provides a discretization of (nonlinear) elasticity.

Explicitly solvable systems provide intuition and include some meaningful physical examples

a) $\ddot{x} = -Ax$ where $A \geq 0$ is a symmetric matrix can be solved by diagonalizing A. (Discrete version of linear wave eqn is a special case)

b) any system with just one degree of freedom can be understood by phase plane analysis; the "planar pendulum" $\ddot{\theta} = -\sin\theta$ is a classic example.

c) The 2 body prbm in $\mathbb{R}^3$

$$m_1 \ddot{x}_1 = \frac{-\partial U}{\partial x_1} \qquad m_2 \ddot{x}_2 = \frac{-\partial U}{\partial x_2}.$$

where $U = U(|x_1 - x_2|)$ can be reduced to a system with 1 deg of freedom (and so fully understood by phase plane analysis). When $U(r) = -k/r$ (the gravitational force potential) this leads to Kepler's laws.

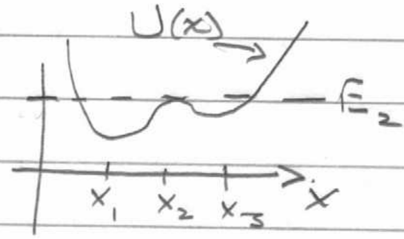
I recommend reading about such examples (Arnold, Landau/Lifshitz, + José/Saletan all have such discussions in 1st chapter or two).

Let me dwell a bit more on (b) = phase plane analysis. The pt is that if $x(t) \in \mathbb{R}$ and $\ddot{x} = -U'(x)$ then eqn determines a flow in 2D "phase plane" $(x, \dot{x})$. Since

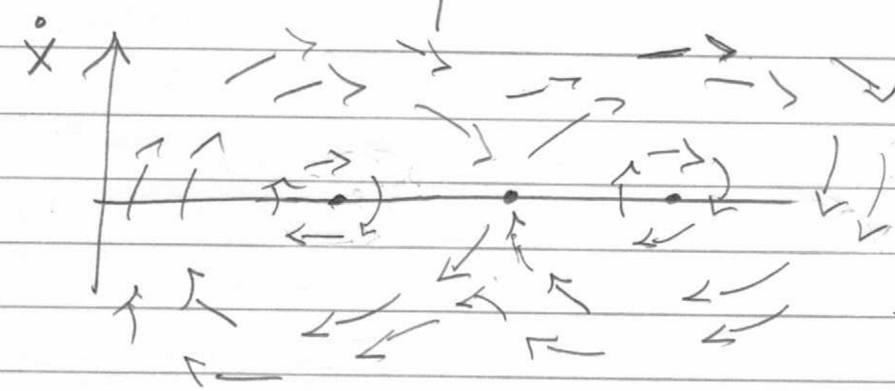
$$H = \tfrac{1}{2}\dot{x}^2 + U(x) \quad \text{is constant,}$$

flow moves along the 1D curve $H = \text{const}$ (which may or may not be closed). For example: if

$$\ddot{x} = -U'(x) \qquad \text{where}$$



Then trajectories move on $\frac{1}{2}\dot{x}^2 + U(x) = E$ (which are closed, in this case). The roots x_1, x_2, x_3 of $U'(x)=0$ are steady states; x_1 + x_3 are stable but x_2 is unstable. Phase plane looks like



For $E > E_2$ (the value of U at x_2) the trajectory surrounds all 3 crit pts.

There are in fact two distinct approaches to this subject.

① The "Hamiltonian viewpt", according to which we're solving a certain class of ODE's

② The "Lagrangian viewpt", according to which our ODE's are the Euler–Lagrange eqns of a variational problem $\int L(x(t), \dot{x}(t))\, dt$

For the systems discussed thus far, of the form

$$m_i \ddot{x}_i = -\frac{\partial}{\partial x_i} U$$

(where each $x_i \in \mathbb{R}^n$ and $U = U(x_1, \cdots, x_N)$)
The Lagrangian perspective is very simple:

$$L(x, \dot{x}) = \frac{1}{2} \sum m_i |\dot{x}_i|^2 - U(x_1, \cdots, x_N)$$

here $x = (X_1, \cdots, X_N) \in \mathbb{R}^{nN}$
$\dot{x} = (\dot{x}_1, \cdots, \dot{x}_N) \in \mathbb{R}^{nN}$

note the minus sign

and the equivalence is elementary: EL eqn is

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)$$

which reduces to

$$-\frac{\partial U}{\partial x_i} = \frac{d}{dt}\left(m_i \dot{x}_i\right) = m_i \ddot{x}_i$$

(I have assumed m_i is constant; otherwise the correct form of Newton's law is not $m_i \ddot{x}_i = -\partial U/\partial x_i$ but rather $\frac{d}{dt}(m_i \dot{x}_i) = -\partial U/\partial x_i$)

Full scope of the theory is easy to describe in Lagrangian setting: we can consider any L that's convex in $\dot{x}$ (with x held fixed), with faster-than-linear growth as $|\dot{x}| \to \infty$.

Full scope of Hamiltonian approach is also easy to describe: we're considering ODE's that can be expressed (in a well-chosen coord system) as

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

For some $H = H(q_1, \dots, q_N; p_1, \dots, p_N)$.
In our example $m_i \ddot{x}_i = -\partial U / \partial x_i$, $p_i = m_i \dot{x}_i$, $q_i = x_i$, and

$$H = \frac{1}{2} \sum \frac{1}{m_i} |p_i|^2 + U(q_1, \dots, q_N)$$

note the plus sign this time

(For a general L, it takes some work to see what p should be. We'll do that later.)

Each viewpt has its own advantages.

Let me dwell a bit now on two key advantages of the Lagrangian viewpt

(A) It's easy to consider constrained problems this way. (Just impose a constraint on the var'l pbm; the EL eqn gets a Lagrange multiplier, of course.)

(B) It's easy to change coords in the Lagrangian framework.

(Please note, however: trajectories are not necessarily minimizers of $\int L(x,\dot{x})\,dt$; rather, in general they are just critical pts.)

Let's spend some time on (A) + (B).

<u>Warmup to (A),(B)</u>: Let's see the analogous assertion in a more familiar setting, namely <u>geodesics</u> (which solve the EL for <u>arclength</u>)

- in $\mathbb{R}^n$, arclength $= \int_{t_1}^{t_2} |\dot{y}(t)|\,dt$; EL eqn (if $\dot{y} \neq 0$) is

$$\frac{d}{dt}\left(\frac{\dot{y}}{|\dot{y}|}\right) = 0.$$

Critical pts are thus straight segments. (Since arclength is indep of parametrization we may choose $|\dot{y}|=1$ and then EL becomes $\ddot{y}=0$.)

- Now what about geodesics on the sphere $|y|=1$? Introducing Lagr mult $g(t)$ for the ptwise constraint $|y(t)|^2=1$, we want

$$\int_{t_1}^{t_2} |\dot{y}| + g(t)\left(|y|^2-1\right)\, dt$$

to be stationary, ie.

$$\frac{d}{dt}\left(\frac{\dot{y}}{|\dot{y}|}\right) = 2g(t)\,y(t)$$

So $\frac{d}{dt}(\dot{y}/|\dot{y}|)$ is normal to the sphere. (Again, we may choose $|\dot{y}|=1$ and then $\ddot{y}$ is normal to the sphere.)

- What if we prefer to use local coords on the sphere? If local coords are $(\xi_1, \dots, \xi_{n-1})$ then our curve $y(t)$ is represented in coords by a curve $\xi(t)$, and

$$|\dot{y}|^2 = \sum g_{ij}(\xi)\, \dot{\xi}_i \dot{\xi}_j$$

where $g_{ij}(\xi)$ (the Riemannian metric) comes from the change of variables by chain rule. So in local coords we're considering crit pts of

$$\int_{t_1}^{t_2} \left(\sum g_{ij}(\xi)\, \dot{\xi}_i \dot{\xi}_j \right)^{1/2} dt$$

Its crit pts are the same curves we found earlier by the method of Lagrange multipliers.

The mechanical analogue of this discussion is almost the same:

- A free particle in space (no forces) is modelled using $L(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2$
  $\Rightarrow$ trajectory is a crit pt of $\int \frac{1}{2}|\dot{x}|^2 dt$
  $\Rightarrow$ $\ddot{x} = 0$ $\Rightarrow$ trajectory is a line.

  Note also: $\frac{d}{dt}|\dot{x}|^2 = 2\langle \dot{x}, \ddot{x} \rangle = 0$ so speed is constant.

  (Same as geometric pblm except speed has to be constant.)

- A marble rolling in a spherical bowl (no gravity!) follows a path that's a crit pt of

$$\int_{t_1}^{t_2} \tfrac{1}{2} |\dot{x}|^2 \, dt \quad \text{constrained by } |x|^2 = 1$$

(if bowl has radius 1). By method of Lagr multipliers

$$\ddot{x} = \text{multiple of } x(t)$$

(ie it is normal to the sphere)

Again, $\frac{d}{dt} |\dot{x}|^2 = 2 \langle \dot{x}, \ddot{x} \rangle = 0$, so speed is constant. We again pick up soln of geometric plm (a geodesic) but parametrized with constant speed.

- In local coords for the sphere, we look instead at EL eqn for

$$\frac{1}{2} \int_{t_1}^{t_2} \sum g_{ij}(\xi) \, \dot{\xi}_i \, \dot{\xi}_j \, dt,$$

(Of course we still get a constant-speed geodesic.)

This discussion used a sphere only to keep the math simple & help you visualize. The same calculation actually works for motion constrained to any surface.

[You might be wondering: if the Lagrangian is $L(\xi, \dot{\xi}) = \frac{1}{2} \sum g_{ij}(\xi) \dot{\xi}_i \dot{\xi}_j$, what is the associated Hamiltonian system? The answer is

$$H(\xi, p) = \frac{1}{2} \langle g^{-1}(\xi) p, p \rangle$$

$$p_i = \sum g_{ij}(\xi) \dot{\xi}_j$$

where g^{-1} is the matrix inverse of g. This is a special case of the correspondence btwn Lagrangian & Hamiltonian settings, to be discussed soon. Of course, the equivalence of the EL for L and Hamilton's eqns can also be checked directly by mere calculation.]