MECHANICS – Problem Set 5, assigned 4/22/19, due 5/8/19

These problems reinforce our discussion of classical mechanics.

(1) Consider a 1D particle with unit mass moving in a potential $U(x)$, in other words the ODE $\ddot{x} = -U'(x)$, whose Hamiltonian is $H = T + U$ with $T = \frac{1}{2}\dot{x}^2$. Suppose the level set $H = E$ is a closed orbit, and let $A(E)$ be the area enclosed by this orbit in the $(x, \dot{x})$ plane. Show that the period of the orbit is then dA/dE.

(2) Consider a particle with unit mass in the plane, which is constrained to stay on the circle $|x| = r(t)$ where r is a fixed function of time. (Aside from this constraint there are no other forces). In polar coordinates, the particle's location is fully determined by $\theta(t)$ (since its distance from the origin $r(t)$ is fixed in advance).

(a) What is the associated Lagrangian variational principle? What ODE must $\theta(t)$ solve?

(b) What is the Hamiltonian description of this mechanical system?

(c) Is the value of H conserved? Why or why not?

(3) (The "principle of least travel time.") Consider an inhomogeneous medium in which the speed of travel at x is $1/f(x)$ (a positive function of location, independent of direction). Our starting point is the observation that for a parametrized path $y(s)$ from $y(s_1) = x_1$ to $y(s_2) = x_2$, the associated travel time is

$$\int_{s_1}^{s_2} f(y(s))|\dot{y}|\,ds.$$

A critical point of this functional is known as a "path of least travel time" (though this terminology is sloppy, since it might be just a saddle point rather than a local minimum of the functional). Consider the mechanical system with Lagrangian $L(x, \dot{x}) = \frac{1}{2}f^2(x)|\dot{x}|^2$. Show that $x(t)$ is a critical point of the associated Lagrangian variational principle

$$\int_{t_1}^{t_2} \tfrac{1}{2}f^2(x(t))|\dot{x}|^2\,dt$$

if and only if (i) it is a path of least travel time, and (2) the path is parametrized so that $f^2(x(t)|\dot{x}(t)|^2$ is constant. (Note: the case $f = 1$ is discussed at the end of the Lecture 8 notes.)

(4) (As promised in the Lecture 9 notes.) Consider the Lagrangian equations associated with

$$L(q, \dot{q}) = \left(\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\dot{q}_i\dot{q}_j\right) - U(q)$$

where $a_{ij}(q)$ is a positive definite symmetric matrix-valued function of $q \in R^N$. Show (by direct differentiation) that if we define $p_i(t) = \sum_{j=1}^{N} a_{ij}(q(t))\dot{q}_j(t)$ then $q(t)$ and $p(t)$ solve Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i},$$

when $H(q,p) = T(q,p) + U(q)$ with

$$T = \frac{1}{2} \sum_{i,j=1}^{N} [a^{-1}]_{ij}(q) p_i p_j.$$

(Note: this is a special case of the general correspondence between the Lagrangian and Hamiltonian formulations of classical mechanics; therefore it is possible to simply specialize the proof of that correspondence to this special case. I am, however, asking for something different: an argument that reaches the conclusion as directly as possible.)