**MECHANICS – Problem Set 4, assigned 3/27/19, due 4/17/19**

These problems are concerned with linear elasticity. Problem 1 asks you to explain why an isotropic Hooke's law is described by just two constants. Problems 2 and 3 explore Korn's inequality. Problem 4 gives an example of an elasticity problem with an explicit separation-of-variables solution. Problems 5-8 examine some important special solutions and reductions of linear elasticity.

1. **Elastic symmetries.** A linearly elastic material is symmetric under a rotation $R$ if its Hookes' law satisfies $\alpha(R^T eR) = R^T \alpha(e)R$. Show, by a direct argument, that if this holds for any $R \in SO(3)$ then $\alpha e = 2\mu e + \lambda(\mathrm{tr}\, e)I$ for some constants $\lambda, \mu$. (Hint: start by showing that $\sigma = \alpha e$ must be simultaneously diagonal with $e$.) What about the case of "cubic symmetry", when $\alpha$ is only symmetric under 90 deg rotations (i.e. under any $R$ which permutes the coordinate axes)?

2. **Korn's inequality for periodic deformations.** Korn's inequality for periodic deformations says
$$\int_Q |\nabla u|^2 \, dx \le C \int_Q |e(u)|^2 \, dx$$
when $u : R^n \to R^n$ is periodic in each variable with period 1 and $Q = [0,1]^n$ is the unit cell. Give a proof using the Fourier representation of $u$. What is the best possible value of the constant $C$? Why is there no condition about $\int \nabla u$ being symmetric?

3. **Korn's inequality for beams.** Let $\Omega_h \subset R^2$ be the long, thin domain $\{0 < x < 1, \, -h/2 < y < h/2\}$ where $h \ll 1$. Korn's second inequality for this domain says
$$\int_{\Omega_h} |\nabla u|^2 \, dx \le C(h) \int_{\Omega_h} |e(u)|^2 \, dx \quad \text{provided } \int_{\Omega_h} \nabla u \text{ is symmetric.}$$

   (a) Show that $C(h)$ must be at least of order $h^{-2}$, by considering deformations of the form $u(x,y) = (-y\phi_x, \phi)$ where $\phi = \phi(x)$.

   (b) Show that the inequality is true with $C_h \sim h^{-2}$. You may assume (for simplicity, this is not really necessary) that $1/h$ is an integer. Hint: divide $\Omega_h$ into $1/h$ squares of side $h$. Korn's inequality (for squares) controls $\nabla u - \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix}$ on the jth square in terms of the strain on that square, for some $\omega_j \in R$. Use Korn's inequality again (this time for rectangles of eccentricity 2) to control $\omega_j - \omega_{j-1}$ in terms of the strain on the (j-1)st and jth squares. Then apply a discrete version of Poincare's inequality in one space dimension to control the variation of $\omega_j$ with $j$.

   (c) How do you think these results would extend to a thin plate-like domain $\{0 < x < 1, \, 0 < y < 1, \, -h/2 < z < h/2\}$ in $R^3$? (Just discuss how the 3D problem is similar or different; I'm not asking for a complete solution.)

4. **Separation of variables.** Let $\Omega$ be a "ball with a hole removed":
$$\Omega = \{x \, : \, \rho^2 < |x|^2 < 1\} \, .$$

Suppose it is filled with an isotropic, homogeneous, linearly elastic material with Lamé moduli $\lambda$ and $\mu$, and constant pressure $P$ is applied at the outer boundary $|x| = 1$. The inner boundary $|x| = \rho$ is traction-free. Find the displacement $u(x)$ and the associated stress $\sigma(x)$ using separation of variables.

5. **The torsion problem.** Let $D$ be a domain in the $x - y$ plane, and consider a long cylinder with cross-section $D$. Imagine twisting the cylinder at its ends. The lateral boundaries are traction-free, and gravity is ignored. The linearized version of such a deformation is achieved by

$$u(x, y, z) = \tau(-yz, xz, \phi(x, y))$$

for $\tau \in R$ and $\phi : D \to R$.

   (a) Find the associated stress and strain, assuming an isotropic and homogeneous Hooke's law. Show that $u$ solves the equations of elastostatics with traction-free boundary condition $\sigma \cdot n = 0$ at the lateral boundaries (and a suitable displacement boundary condition at the ends) if and only if $\Delta\phi = 0$ in $D$ and $\partial\phi/\partial n = (y, -x) \cdot n$ at $\partial D$.

   (b) Verify that the consistency condition $\int_{\partial D}(y, -x) \cdot n = 0$ is satisfied [ thus $\phi$ exists and is unique up to an additive constant].

   (c) Show that the elastic energy per unit length is $\tau^2 T$ where $T = \mu \int_D (\phi_x - y)^2 + (\phi_y + x)^2 \, dxdy$. This $T$ is called the *torsional rigidity* of the cylinder.

[Comment: This example is more than just a special solution: "Saint Venant's principle" says that no matter how you twist the ends of a cylinder, far from the ends the deformation will approach the special solution described above.]

6. **Antiplane shear.** Consider once again a cylinder with cross-section $D$, but consider a uniform body load in the $z$ direction (gravity), and suppose the lateral boundaries are clamped. Show that these conditions are consistent with the displacement $u = (0, 0, \phi(x, y))$ with $\Delta\phi = 1$ in $D$ and $\phi = 0$ at $\partial D$.

7. **Bending of a thin plate.** Consider now a thin, constant-thickness plate whose midplane occupies a region $D$ in the $x-y$ plane. The upper and lower surfaces are $z = \pm h/2$, so the thickness is $h$. Consider a deformation of the form $u = (-z\phi_x, -z\phi_y, \phi + \frac{\alpha}{2} z^2 \Delta\phi)$. Find the associated strain and stress, keeping only terms of order $h$. Show that for the faces to be traction-free (to this order) we need $\alpha = \lambda/(\lambda + 2\mu)$. Do the $z$- integrations in the basic variational principle, to obtain a new variational principle for $\phi(x, y)$. Notice that it involves second derivatives of $\phi$, so the associated PDE is a fourth-order equation!

8. **Plane stress.** Consider the same thin plate, but rather than bending it we suppose it is loaded within its plane. The top and bottom are traction-free, so $\sigma_{i3} = 0$ there. If the plate is thin enough we may expect that $\sigma$ is independent of $z$. This does not imply that $u_i$ are independent of $z$, but we can nevertheless consider $\bar{u}_i(x, y)$ =the average of $u_i$ with respect to $z$. Show that $\bar{u}_1, \bar{u}_2$ solve the system of "2D elasticity" with a suitable choice of elastic constants.