

Mechanics - Handout on Korn's 2nd inequality

On Korn's 2nd \neq , namely:

$$(1) \quad \int_{\Omega} \nabla_i u_j \text{ symmetric } \forall i, j \Rightarrow \int_{\Omega} |\nabla u|^2 \leq C \int_{\Omega} |e(u)|^2$$

Start is true for any "Lipschitz domain" ($\partial\Omega$ locally a Lipschitz graph, with Ω lying to one side). The following proof is essentially Oleinik + Koldratov's CRAS note, but simplified slightly by hypothesis that $\partial\Omega$ is C^1 rather than Lipschitz.

Step 1: Observe it suffices to prove

$$(2) \quad \int_{\Omega} |\nabla u|^2 \leq C \left[\int_{\Omega} |e(u)|^2 + \int_{\Omega} |u|^2 \right]$$

ie to show $H^1(\Omega)$ norm is equiv to $\left(\int_{\Omega} |e(u)|^2 + |u|^2 \right)^{1/2}$.
[Proof that (2) \Rightarrow (1) can be used for many analogous pbms.]

Argue by contradiction: if (2) holds but (1) fails then
 \exists seq $u^n \rightarrow t$

$$\int_{\Omega} u^n = 0, \quad \int \nabla_i u_j^n \text{ symmetric}$$

$$\int_{\Omega} |\nabla u^n|^2 = 1, \quad \int |e(u^n)|^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Korn's inequality $\mathbb{P}S^2$

Sequence is bdd in H^1 (here we use the inequality $\int_{\Omega} |u - \bar{u}|^2 \leq C \int_{\Omega} |\nabla u|^2$) so \exists subseq. converging wklly in H^1 and strongly in L^2 to a limit u^* .
 May suppose wlog that whole seq. converges to u^* .

Evidently $e(u^*) = 0$. Applying (2) to $u^n - u^*$ gives

$$\int_{\Omega} |\nabla(u^n - u^*)|^2 \rightarrow 0$$

so actually $u^n \rightarrow u^*$ strongly in H^1 . In particular

$$\int_{\Omega} |\nabla u^*|^2 = \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u^n|^2 = 1$$

But

$$\int_{\Omega} \nabla_i u_j^* = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla_i u_j^n \quad \text{is symmetric}$$

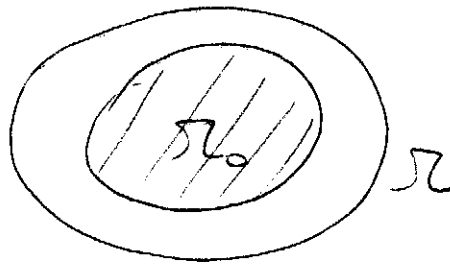
$$\int_{\Omega} u_i^* = \lim_{n \rightarrow \infty} \int_{\Omega} u_i^n = 0$$

and the last two relns, with $e(u^*) = 0$, imply $u^* \equiv 0$.
 This contradicts $\int_{\Omega} |\nabla u^*|^2 = 1$. So (1) holds after all.

Step 2: Observe it suffices to show, for some $\Omega_0 \subset\subset \Omega$

$$(3) \quad \int_{\Omega} |\nabla u|^2 \leq C \left[\int_{\Omega} |e(u)|^2 + \int_{\Omega_0} |\nabla u|^2 \right]$$

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Indeed, suppose (3) is known. Then choose φ st

$$\varphi \equiv 1 \quad \text{on } \Omega_0$$

$$\varphi = 0 \quad \text{near } \partial\Omega$$

and apply Korn's 1st \neq to $u\varphi$:

$$\begin{aligned} \int_{\Omega_0} |7u|^2 &\leq \int_{\Omega} |7(u\varphi)|^2 \\ &\leq C \int_{\Omega} |e(u\varphi)|^2 \\ &\leq C \left[\int_{\Omega} |e(ux)|^2 + |u|^2 \right] \end{aligned}$$

Step 3: Preparing to prove (3), we seek a decomposition $u = w + (u-w)$ st.

- $\int_{\Omega} |7w|^2 \leq C \int_{\Omega} |e(ux)|^2$, so w is not problematic
- $\Delta(u-w) = 0$ on Ω , so we can use PDE to estimate $u-w$.

Main tool is the identity

$$(4) \quad \frac{\partial^2 v_i}{\partial x_p \partial x_q} = \frac{1}{2} \left[\frac{\partial}{\partial x_p} e_{iq}(v) + \frac{\partial}{\partial x_q} e_{ip}(v) - \frac{\partial}{\partial x_i} e_{pq}(v) \right]$$

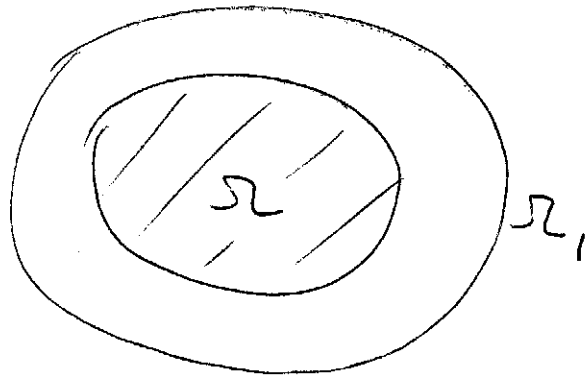
which gives

$$\Delta u_i = \sum_p \frac{\partial}{\partial x_p} e_{ip}(u) - \frac{1}{2} \frac{\partial}{\partial x_i} e_{pp}(u)$$

So a convenient construction is

$$\begin{aligned} \Delta W &= \operatorname{div} \tilde{\sigma} && \text{in } \Omega_1 \\ W &= 0 && \text{at } \partial\Omega_1 \end{aligned}$$

where $\Omega \subset \subset \Omega_1$



and

$$\tilde{\sigma}_{ij} = \begin{cases} e_{ij} - \frac{1}{2} (\operatorname{tr} e) \delta_{ij} & \text{in } \Omega \\ 0 & \text{in } \Omega_1 \setminus \Omega \end{cases}$$

This works since

$$\int_{\Omega_1} |\nabla W|^2 = - \int_{\Omega_1} W \operatorname{div} \tilde{\sigma} = \int_{\Omega_1} \langle \nabla W, \tilde{\sigma} \rangle \leq \left(\int_{\Omega_1} |\nabla W|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\sigma|^2 \right)^{\frac{1}{2}}$$

whence

$$\int_{\Omega} |\nabla w|^2 \leq \int_{\Omega_1} |\nabla w|^2 \leq \int_{\Omega} |\sigma|^2 \leq C \int_{\Omega} |e(z)|^2$$

Step 4: Estimate $z = u - w$, taking advantage of fact that $\Delta z = 0$ in Ω (though we have no useful bc at $\partial\Omega$). We do this using

Lemma A: Let $\rho(x) = \text{dist}(x, \partial\Omega)$ and suppose $\Delta v = 0$ in Ω .
Then

$$\int_{\Omega} \rho^2 |\nabla v|^2 \leq C \int_{\Omega} v^2$$

Lemma B: Choosing $\Omega_0 \subset\subset \Omega$ conveniently, every f satisfies

$$\int_{\Omega \setminus \Omega_0} f^2 \leq C \left[\int_{\Omega} \rho^2 |\nabla f|^2 + \int_{\Omega_0} f^2 \right]$$

(Proofs postponed to step 5).

Applying Lemma A to $v = e_{ij}(z)$ (note: $\Delta z_i = 0 \forall i \Rightarrow \Delta e_{ij}(z) = 0$) gives

$$\int_{\Omega} \rho^2 |\nabla e(z)|^2 \leq C \int_{\Omega} |e(z)|^2$$

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Recall from (4) that $\nabla \nabla z$ is representable in terms of $\nabla e(z)$, so we conclude

$$(5) \quad \int_{\Omega} \rho^2 |\nabla \nabla z|^2 \leq C \int_{\Omega} |e(z)|^2$$

Now apply Lemma B to $f = \nabla z$, and combine with (5), to see

$$\begin{aligned} \int_{\Omega \setminus \Omega_0} |\nabla z|^2 &\leq C \left[\int_{\Omega} \rho^2 |\nabla \nabla z|^2 + \int_{\Omega_0} |\nabla z|^2 \right] \\ &\leq C \left[\int_{\Omega} |e(z)|^2 + \int_{\Omega_0} |\nabla z|^2 \right] \end{aligned}$$

Thus evidently

$$(6) \quad \int_{\Omega} |\nabla z|^2 \leq C \left[\int_{\Omega} |e(z)|^2 + \int_{\Omega_0} |\nabla z|^2 \right]$$

Now we're essentially done; in fact $|\nabla u|^2 \leq 2(|\nabla w|^2 + |\nabla z|^2)$ by triangle inequality, and

$$\int_{\Omega} |\nabla w|^2 \leq C \int_{\Omega} |e(w)|^2 \quad \text{by step 3}$$

while

$$\int_{\Omega} |\nabla z|^2 \leq C \left[\int_{\Omega} |e(w)|^2 + \int_{\Omega_0} |\nabla u|^2 \right]$$

by (6) and several easy applications of triangle inequality.
(remember: $u = z + w \Rightarrow z = u - w$)

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whence $\int_{\Omega} |\nabla u|^2 \leq C \left[\int_{\Omega} |\operatorname{curl} u|^2 + \int_{\Omega_0} |\nabla u|^2 \right]$ as desired.

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Step 5 Proofs of Lemmas A+B.

Pf of Lemma A: $\Delta v = 0 \Rightarrow \rho^2 v \Delta v = 0 \Rightarrow$

$$\operatorname{div}(\rho^2 \nabla v) = 2 \rho \nabla \rho \cdot \nabla v + \rho^2 |\nabla v|^2$$

$$\Rightarrow \int \rho^2 |\nabla v|^2 \leq 2 \int \rho |\nabla \rho| |v| |\nabla v|$$

Now $|\nabla \rho| = 1$, and $2ab \leq \delta a^2 + \frac{1}{\delta} b^2$, so for any $\delta > 0$ we have

$$\int_{\Omega} \rho^2 |\nabla v|^2 \leq \delta \int_{\Omega} \rho^2 |\nabla v|^2 + \frac{1}{\delta} \int_{\Omega} v^2$$

Choosing $\delta < 1$ gives

$$\int_{\Omega} \rho^2 |\nabla v|^2 \leq C \int_{\Omega} v^2$$

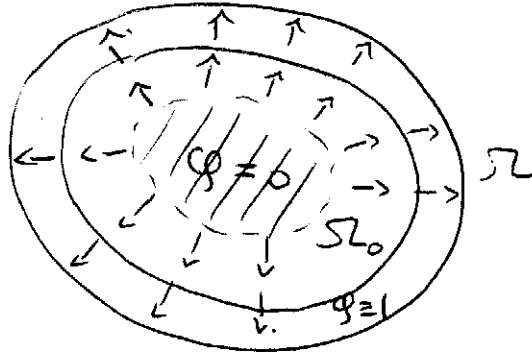
as desired.

Pf of Lemma B: (Here is where we simplify Kondratiev-Oleinik by assuming $\partial\Omega$ is smooth enough). Assume $\vec{\xi} = \nabla \rho$ is C^1 in a nbhd of $\partial\Omega$. Choose cutoff φ , and subdomain $\Omega_0 \subset \subset \Omega$, st.

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$\varphi \equiv 1$ on $\Omega \setminus \Omega_0$

$\varphi \equiv 0$ where $\vec{\xi} = \nabla \rho$ is not C^1



We have

$$\operatorname{div} (f^2 \rho \varphi \vec{\xi}) = 2f(\nabla f \cdot \vec{\xi}) \rho \varphi + f^2 (\nabla \rho \cdot \vec{\xi}) \varphi + f^2 \rho \varphi \operatorname{div} \vec{\xi} + f^2 \rho \nabla \varphi \cdot \vec{\xi}$$

Making Ω_0 larger if necessary, we may suppose that

$$|\rho \operatorname{div} \vec{\xi}| \leq \frac{1}{2} \quad \text{on } \Omega \setminus \Omega_0$$

and we know of course $\nabla \rho \cdot \vec{\xi} \equiv 1 + \varphi \equiv 1$ on $\Omega \setminus \Omega_0$.

So we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega \setminus \Omega_0} f^2 &\leq \int_{\Omega} 2f(\nabla f \cdot \vec{\xi}) \rho \varphi + \int_{\Omega} f^2 \rho \nabla \varphi \cdot \vec{\xi} \\ &\quad + \int_{\Omega_0} f^2 (\rho \varphi \operatorname{div} \vec{\xi} + (\nabla \rho \cdot \vec{\xi}) \varphi) \end{aligned}$$

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$$\leq \delta \int_{\Omega} f^2 + \frac{1}{\delta} \int_{\Omega} \rho^2 |\nabla \mathbf{u}|^2 + C \int_{\Omega_0} f^2$$

Choosing δ small again we get

$$\int_{\Omega \cup \Omega_0} f^2 \leq C \left[\int_{\Omega} \rho^2 |\nabla \mathbf{u}|^2 + \int_{\Omega_0} f^2 \right]$$

(with a new choice of C), as desired.