Mechanics - Lecture 2+3, 2/6/2019 & 2/13/2019

Today's topic (spilling most surely to part if not all of next week): a 1D bending theory (Euler's "elastica"), both as 1st exposure to "torque" and "bending", and as an example of bifurcation.

Antman's chapter 4 has a good discn of beams + rods; his chapter 5 provides an introduction to bifurcation. For something shorter (along the lines of these notes) see §4.9 of Howell-Kozyreff-Ockendon.

Big picture:

- in 2D, there is resistance to bending even for a sheet that's inextensible (eg paper, cf the "xerox paper problem" discussed later in these notes + explored in HW2).

  Essential mechanism: if a thin strip is wrapped to an annulus with midline wrapping isometrically, then lines parallel to midline are stretched or shrunk due to effects of curvature

- for a rod in 3D (or a thin ribbon - which is just a rod with rectangular cross-section) picture is similar except that rod can <u>bend</u> (in two possible dirns) + <u>twist</u> (example: a Mobius band)

To keep things simple we'll focus on an <u>inextensible</u> <u>beam</u> (initially straight + uniform, eg a piece of paper or a ruler, deformed by bending consistent with a 1D model).

To get started, must discuss:

a) <u>kinematics</u> (description of deformation)

b) <u>statics</u> (forces + bending moments, and assoc balance laws)

c) <u>constitutive laws</u> (in this case: relation between curvature + bending moment)

<u>Kinematics</u>: for a 1D inextensible rod this is easy: use $0<s<L$ as reference body, and $\vec{r}(s)$ as deformed position. We want to assume (for simplicity) $\vec{r}(s)$ stays in the x_1-x_2 plane, so $|\vec{r}_s|=1 \Rightarrow$

$$\vec{r}_s = (\cos\theta(s), \sin\theta(s))$$

The rod's <u>curvature</u> is $\theta'(s)$. (This will play a role similar to that of the "strain" $|\vec{r}_s| - 1$ in our discn of strings.)

<u>statics</u>: slicing the rod at $s = s_0$, each side acts on the other by

(i) a net force $\vec{n}(s)$ (as with strings)

(ii) an additional bending moment $\vec{m}(s)$

(Think of a curved piece of paper, ignoring gravity:  viewed as a 1D rod; here $\vec{n} = 0$, and $\vec{m}(s)$ is what maintains the bent state.)

Defn of $\vec{m}$: part of beam at $s > s_0$ exerts <u>torque</u> $\vec{r}(s_0) \times \vec{n}(s_0) + \vec{m}(s_0)$ on the rest. Note that for deformations in the x_1, x_2 plane, $\vec{m} = (0, 0, M(s))$ is essentially scalar-valued.

Balance of forces: $\vec{n}(s_1) - \vec{n}(s_0) + \int_{s_0}^{s_1} \vec{f}(s)\, ds = 0$

Balance of torques: $\left[\vec{m}(s_1) + \vec{r}(s_1) \times \vec{n}(s_1)\right] - \left[\vec{m}(s_0) + \vec{r}(s_0) \times \vec{n}(s_0)\right]$
$+ \int_{s_0}^{s_1} \vec{r}(s) \times \vec{f}(s)\, ds = 0$

whence $\quad \vec{n}_s + \vec{f} = 0$

$$\vec{m}_s + (\vec{r} \times \vec{n})_s + \vec{r} \times \vec{f} = 0$$

or equivalently:

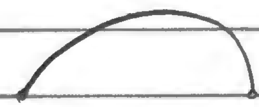
$$\boxed{\begin{aligned} \vec{n}_s + \vec{f} &= 0 \\ \vec{m}_s + \vec{r}_s \times \vec{n} &= 0 \end{aligned}}$$

(Where does balance of torque come from? Well, dynamic version of balance of forces is cons of linear momentum, so it shouldn't be surprising that dynamic version of balance of torque is conservation of angular momentum. This will be clearer when we discuss Classical Mechanics; for now please take the balance laws as a starting pt.)

Notes: (1) if $f = 0$ then $\vec{n}$ is constant (and clearly evident from the bdry conds)

(2) a 1D rod can be held in a bent posn. either by applying forces (horizontal, in the picture below) or by applying bending

moments at the ends



Constitutive law: since the rod is inextensible there is no constitutive law for $\vec{n}$ (instead we get it by integr of $\vec{f}$, with constants of integr coming from bdry conds). Note that $\vec{n}$ need not be in the direction of $\vec{r}_s$.

The simplest law for $\vec{m}$ is the "physically linear" law

$$\vec{m} = (0,0,M), \quad M = A\theta' \qquad A = \text{constant}$$

One can of course consider more nonlinear laws (taking M to be a nonlin fn of θ'). But this linear law is rich enough that we'll stick with it. (The linear law is in fact appropriate for thin bodies, since substantial curvature still means rel little stretching or shrinking of surfaces parallel to the midline.)

When $f=0$ this leads us to the "elastica" (considered by Euler in 1727, but also by

J Bernoulli in 1694; Antman has lots on the history):

$$\vec{r}_s = (\cos\theta(s), \sin\theta(s))$$

$$\vec{n}_s = 0 \Rightarrow \vec{n} = \text{const} = -\Lambda(\cos\alpha, \sin\alpha)$$
for some Λ, α

$$\vec{m}_s + \vec{r}_s \times \vec{n} = 0, \quad \vec{m} = (0,0,M) \Rightarrow$$

$$(0,0,M_s) = \Lambda(\cos\theta, \sin\theta, 0) \times (\cos\alpha, \sin\alpha, 0).$$

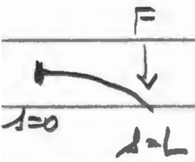
$$\Rightarrow \quad M_s = \Lambda(\cos\theta \sin\alpha - \cos\alpha \sin\theta)$$

$$\Rightarrow \quad M_s = \Lambda \sin(\alpha - \theta)$$

Constitutive law says $M_s = (A\theta'(s))'$. So ODE becomes (writing $\gamma = \theta(s) - \alpha$)

$$[A\gamma'(s)]' + \Lambda \sin\gamma(s) = 0.$$

<u>Note:</u> in general Λ and α are unknown, just like $\gamma(s)$; they must be determined from bdry data.



Example: deflection of a diving board (ignoring gravity). Take $s=0$ to be clamped horizontally ($\theta(0)=0$) + let downward force F be applied at RH edge $s=L$, which is otherwise free ($\theta'(L)=0$). Then $\vec{n}=(0,-F)$ and

$$(0,0,M_s) = (\cos\theta, \sin\theta, 0) \times (0,F,0)$$
$$= F\cos\theta$$

so

$$A\theta_{ss} = F\cos\theta \qquad 0<s<L.$$
with $\theta(0)=0$ and $\theta_s(L)=0$.

A more or less exact soln is possible:

$$A\,\theta_{ss}\,\theta_s = F\,\theta_s\cos\theta$$

$$\Rightarrow \qquad \theta_s^2 = \frac{2F}{A}\sin\theta + \text{const}$$

$$= \frac{2F}{A}\left[\sin\theta + \sin\alpha\right] \qquad \alpha = -\theta(L) > 0$$

Since $d\theta/ds$ should be negative, a bit of arithmetic gives

$$\int_0^{-\theta(s)} \frac{d\varphi}{\sqrt{\sin\alpha - \sin\varphi}} = s\sqrt{\frac{2F}{A}}.$$

The value of α is determined by the eqn

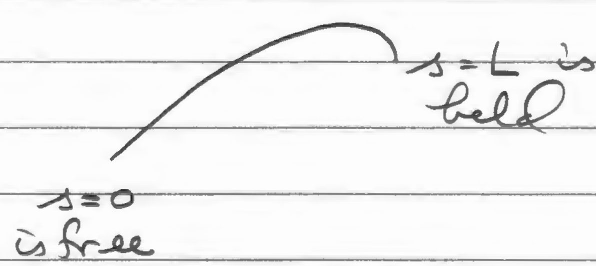
$$\int_0^\alpha \frac{d\varphi}{(\sin\alpha - \sin\varphi)^{1/2}} = L\sqrt{2F/A}$$

(which has no explicit soln but can easily be solved numerically).

Another example: the "xerox paper problem".
Describe the profile of a standard 8.5×11 sheet of paper, held at one edge so the tangent there is vertical.

Differences from the elastica:
- gravity matters
- must specify & use a different bc

Now $\vec{n}_s + f = 0$, $f = f_0(0,-1,0)$; usual conventions



the bc's are
- no force or moment at $s=0$
- specified angle $\theta = -\pi/2$ at $s=L$

Evidently $\vec{n} = (a, b+f_0 s, 0)$ for constants a, b
& bc at $s=0$ give $a=b=0$
$\Longrightarrow \vec{n} = (0, f_0 s, 0)$

Now eqn for $\vec{m}$ + linear constit law gives

$$A\theta'' + f_o s \cos\theta(s) = 0 \qquad 0 < s < L.$$
$$\theta'(0) = 0, \quad \theta(L) = -\pi/2$$

(On HW2 you'll be asked to estimate the value of A/f_o for a piece of xerox paper.)

What can we do with this that's interesting? My choice: use it as an intuitive, physically natural example of bifurcation.

(Where to read more? Antman's Chapter 5 is pretty good – you'll find versions of all that I do there, and much more. Howell-Kozyreff-Ockendon has a concise treatment in §4.9.3. Another good source: Ivar Stakgold, "Branching of solutions of nonlinear eqns" SIAM Review 13 (1971) 289-332.)

Goal: consider the elastica with compressive load λ. If λ is large enough it will buckle. What is the critical load λ_0? How can we understand the buckled configurations?

Various choices of bc are possible; let's choose

$\theta(0) = 0$ and LHS is <u>clamped</u> (so $\vec{r}(0)$ is fixed, and $\vec{r}_s(0) = (1,0)$)

$\theta'(1) = 0$ RHS is "pinned" to loading device (applied load is $(\lambda, 0)$ but applied bending moment is 0)

Eqn (derived earlier) is

$$(*) \qquad \frac{d}{ds}(A\theta_s) + \lambda \sin\theta(s) = 0 \qquad 0<s<1$$

$$\theta(0) = 0, \quad \theta'(1) = 0$$

Clearly $\theta \equiv 0$ is a soln for any λ. When λ is small we expect it to be stable; when λ is large enough we expect it to be unstable.

<u>Note</u>: this pbm (with λ fixed) has a var'l formulation: $\theta(s)$ is a critical pt of

$$E = \int_0^1 \tfrac{1}{2} A \theta_s^2 + \lambda \cos\theta \; ds.$$

subject to $\theta(0) = 0$

(The condn $\theta'(1) = 0$ arises as a "natural" b.c. at $s=1$.) Interpret this as

$$E = (\text{elastic energy}) + \begin{pmatrix}\text{work done by} \\ \text{load}\end{pmatrix}$$

since

$$\int_0^1 \tfrac{1}{2} A \theta_s^2 = \text{"energy" due to curvature}$$

and

$$\int_0^1 \lambda \cos\theta = \lambda \int_0^1 \vec{r}_s \cdot (1,0)\, ds = \vec{r}(1) \cdot (\lambda, 0)$$

is force $\cdot$ (displacement of loaded pt).

Natural physical expt is to increase λ gradually, starting from 0. Amounts mathematically to a "continuation method" for obtaining solns $\theta = \theta(s,\lambda)$. Diffn of eqn wrt λ gives eqn for $\dot\theta = \partial\theta/\partial\lambda$, which we can expect to integrate (an ode in λ) to get θ. This procedure is especially simple in the given example: diffn of $(*)$ wrt λ gives

$$(**) \qquad (A\dot{\theta}_s)_s + \lambda(\cos\theta)\,\dot{\theta} + \sin\theta = 0$$

$$\dot{\theta}(0) = 0, \quad \dot{\theta}_s(1) = 0$$

If $\theta(s) \equiv 0$ then (**) implies $\dot{\theta}(s) \equiv 0$ so long as $\lambda <$ 1st eigenvalue of linearized pblm

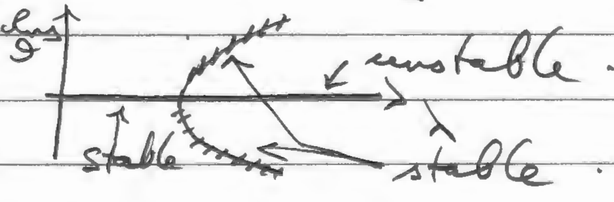
$$A\dot{\theta}_{ss} + \lambda\dot{\theta} = 0, \quad \dot{\theta}(0) = \dot{\theta}_s(1) = 0$$

From now on let's take <u>$A = 1$</u> for simplicity. Then 1st eigenvalue is

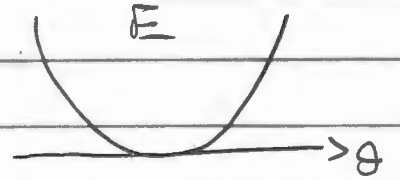
$$\lambda_1 = \pi^2/4, \quad \text{assoc to eigenfunction } \varphi(s) = \sin\left(\tfrac{\pi}{2}s\right)$$

Conclusion so far: when $A=1$, crit load is $\pi^2/4$. (For general $A>0$, crit load would be $\pi^2 A/4$ by same argument.)

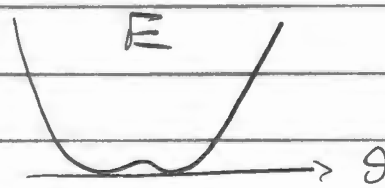
<u>But</u>: expt doesn't stop at $\lambda = \lambda_0$, and neither should we. However we need a viewpt that permits $\theta = \theta(s,\lambda)$ to be nonunique. In fact, we'll show that the <u>bifurcation diagram</u> is (locally, near $\lambda = \lambda_0$) like this:



Variational perspective: for $\lambda > \lambda_0$ the var'l problem has a saddle pt + 2 (nearby) local min



$\lambda < \lambda_0$



$\lambda > \lambda_0$

$\theta \equiv 0$ is a saddle pt!

Rigorous procedure for analysis uses "Lyapunov-Schmidt reduction" (I'll sketch it later). But situation can be captured very concretely by the following more elementary calculation (see Antman's §5.6 or Howell et al §4.9.3): try ansatz

$$\lambda(\varepsilon) = \frac{\pi^2}{4} + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \cdots$$

$$\theta(s,\varepsilon) = 0 + \varepsilon \theta_1(s) + \varepsilon^2 \theta_2(s) + \cdots$$

and expand in powers of ε. Full eqn is

$$0 = (\varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots)'' + \left(\frac{\pi^2}{4} + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \cdots\right)\sin(\varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots)$$

and we have (using $\sin x \simeq x - \frac{1}{6}x^3 + \cdots$)

$$\sin(\varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots) = \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3(\theta_3 - \tfrac{1}{6}\theta_1^3) + \cdots$$

So we get:

<u>at order ε</u> $\quad \theta_1'' + \frac{\pi^2}{4}\theta_1 = 0, \quad \theta_1(0) = \theta_1'(1) = 0$

$$\Rightarrow \theta_1(s) = g\,\varphi(s) \qquad \varphi = \sin(\tfrac{\pi}{2}s)$$

g = any constant

<u>at order ε^2</u> $\quad \theta_2'' + \frac{\pi^2}{4}\theta_2 = -\alpha_1\theta_1 = -\alpha_1 g\,\varphi(s)$

$$\theta_2(0) = 0, \quad \theta_2'(1) = 0$$

Soln exists iff RHS $\perp$ null-vector of LHS, ie if $\int_0^1 \alpha_1 g\,\varphi^2(s)\,ds = 0$. So (assuming $g \neq 0$) $\alpha_1 = 0$, and θ_2 is again a multiple of $\varphi(s)$.

<u>at order ε^3</u> $\quad \theta_3'' + \frac{\pi^2}{4}\theta_3 = -\alpha_2\theta_1 - \underbrace{\cancel{\alpha_1\theta_2}}_{0} + \frac{\pi^2}{4}\cdot\frac{1}{6}\theta_1^3$

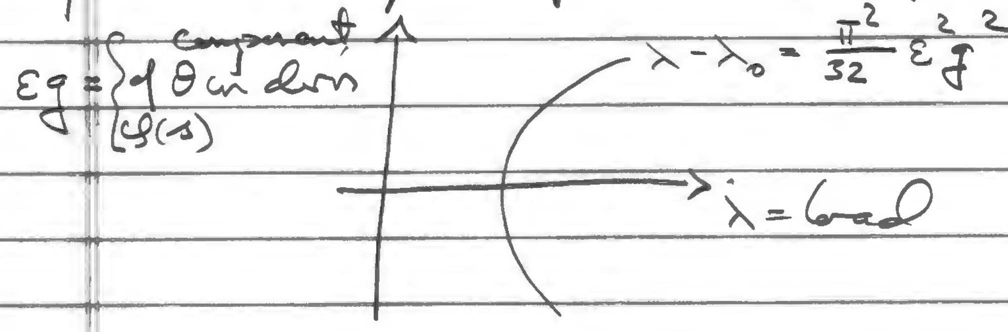
Soln exists iff

$$\int_0^1 -\alpha_2 g\varphi^2 + \frac{\pi^2}{24}g^3\varphi^4\,ds = 0.$$

This simplifies to $\quad \alpha_2 g = \frac{\pi^2}{32}g^3$

since $\int_0^1 \sin^2(\frac{\pi}{2}s) = \frac{1}{2}, \quad \int_0^1 \sin^4(\frac{\pi}{2}s) = 3/8.$

We could continue, but there's no need: we've shown that bifn diagram is locally a parabola, opening to the right (since $\frac{\pi^2}{32} > 0$)



(Bifurcation is called "supercritical" because the parabola opens to the right.)

Here's a sketch of how Liapunov-Schmidt reduction works in this case (see eg Stakgold for more detail):

Let's look for $\theta = g\varphi + \tilde{\theta}$ $\quad \tilde{\theta} \perp \varphi$

where φ = 1st eigenfn $= \sin(\frac{\pi}{2}s)$. Write eqn $\theta'' + \lambda \sin\theta$ as

$$\theta'' + \lambda_0 \theta + (\lambda - \lambda_0)\theta + \lambda(\sin\theta - \theta) = 0$$

ie

(1) $$\theta'' + \lambda_0 \theta = -(\lambda - \lambda_0)\theta - \lambda(\sin\theta - \theta)$$

Consistency condn is

(2) $\quad (\lambda-\lambda_0)\theta + \lambda(\sin\theta - \theta) \perp \varphi$

If this holds, there's a unique $\tilde\theta \perp \varphi$ solving (1). So we can view $\tilde\theta = \tilde\theta_{\lambda,g}$ as being defined (for λ near λ_0 and g near 0) by

(3) $$\tilde\theta'' + \lambda_0\tilde\theta = P_{\varphi^\perp}\left[-(\lambda-\lambda_0)\theta - \lambda(\sin\theta - \theta)\right]$$

$$\tilde\theta \perp \varphi, \quad \tilde\theta(0) = \tilde\theta'(1) = 0.$$

The eqn (2) gives the reln between g + λ that describes the bifn diagram. One can show (using $\sin\theta - \theta \sim -\frac{1}{6}\theta^3$) that $\|\tilde\theta\| \le C|g|^3$, so the leading order character of bifn relation is

$$\int (\lambda-\lambda_0)(g\varphi + \underset{0}{\cancel{\tilde\theta}})\varphi + \lambda_0\left(-\tfrac{1}{6}g^3\varphi^3\right)\varphi = 0$$

ie

$$(\lambda-\lambda_0)\,g\int\varphi^2\,ds - \tfrac{1}{6}\lambda_0 g^3\int\varphi^4\,ds = 0.$$

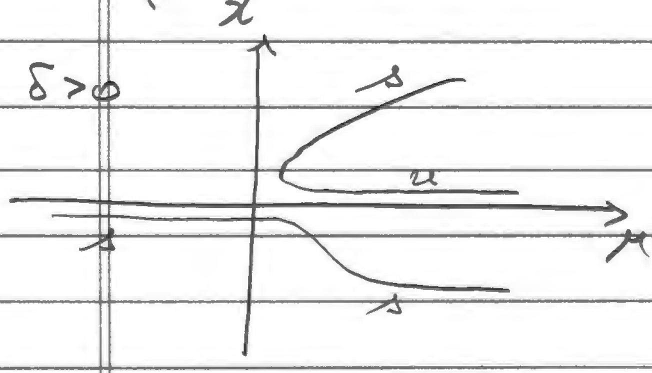
as we obtained earlier by expansion. (Essence of this approach: $\theta = g\varphi + \tilde\theta$ represents the nontrivial solutions as a <u>graph</u> over the 1D axis $g\varphi$.).

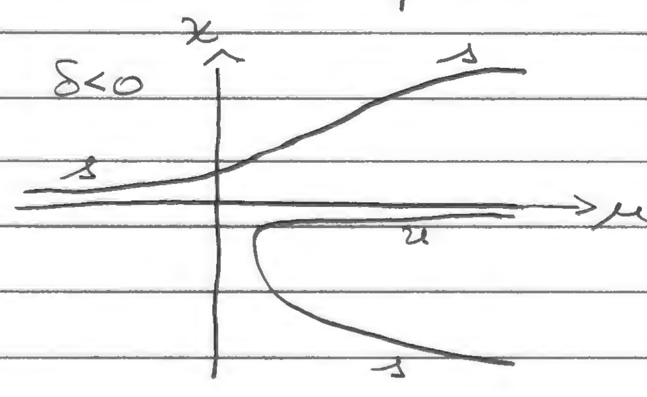
Imperfection sensitivity : an important aspect of bifurcation is that a small imperfection in the system will (typically) change the bifurcation diagram in an essential way. (This is important, because real systems are usually imperfect.)

Simplest example: let's perturb eqn $x^3 = \mu x$ (where $x, \mu \in \mathbb{R}$; note that it describes the crit pts of $\frac{1}{4}x^4 - \frac{\mu}{2}x^2$, which is convex for $\mu < 0$ [$x=0$ is the only crit pt] but not for $\mu > 0$ [crit pts are $x = \pm\sqrt{\mu}$ and $x = 0$]) by considering

$$x^3 - \mu x + \delta = 0$$

(which describes crit pts of $\frac{1}{4}x^4 - \frac{\mu}{2}x^2 + \delta x$). The presence of $\delta \neq 0$ "breaks" the bifn diagram into two disconnected components





(The labels "s" and "u" correspond to local min [stable] or non-min-crit pt [u] of assoc "energy" $\frac{1}{4}x^4 - \frac{\mu}{2}x^2 + \delta x$, with $\mu + \delta$ held fixed. Since x is one-dimensional the non-min crit pt is a local max.)

<u>Note</u>: effect of $\delta \neq 0$ involves fractional powers of δ (Thus: effect is not so small, even if δ is small) since

$$\mu = 0 \implies x^3 = -\delta \implies x = (-\delta)^{1/3}$$

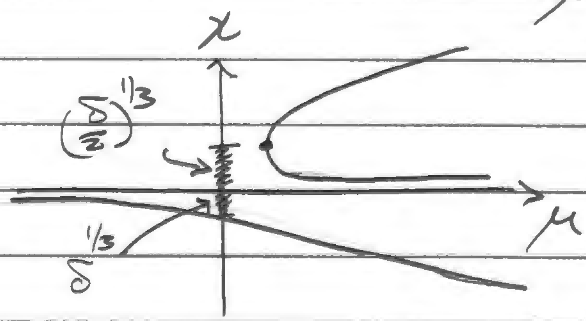
Also: vertical tangent in bif diagram

$$\iff d\mu/dx = 0$$

$$\iff 3x^2 = \mu \iff x = (\mu/3)^{1/2}$$

$$\mu^{3/2} = \frac{3\sqrt{3}}{2}\delta$$

$\delta > 0$



A problem on HW2 asks you to show that the buckling of an elastica with a bit of intrinsic curvature is essentially the same as this example. (Gravity provides another physically natural imperfection; see

Howell - Kozyreff - Ockendon Chap 4, prob 4.17.)

<u>Digression</u>: in the late 70's people asked whether we can "classify all possible was an imperfection can break the bifurcation diagram". Answer was yes in many cases; see Golubitsky + Schaeffer, "A theory for imperfect bifurcation via singularity theory" CPAM 32 (1979) 21-98.