Mechanics – Lecture 7, 3/13/2019

Recall from Lecture 6: for linear elasticity, equil eqn is

$$\operatorname{div}(\alpha e(u)) + f = 0 \quad \text{in } \Omega$$

where α is the Hooke's law (a pos def linear map from symmetric matrices to symmetric matrices). Typical bc are

$$u = u_0 \quad \text{on } \partial\Omega \qquad (\text{"Displacement bc"})$$

or

$$(\alpha e(u)) \cdot n = g \quad \text{at } \partial\Omega \qquad (\text{"Traction bc"}).$$

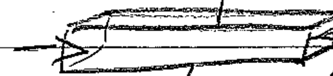
or maybe fix $u = u_0$ on part of $\partial\Omega$ and fix $(\alpha e(u)) \cdot n$ on the rest of $\partial\Omega$.

I'll discuss

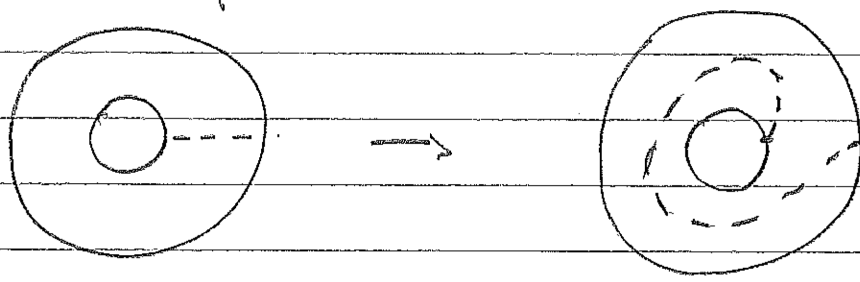
a) uniqueness

b) existence

c) convergence of a basic Galerkin approxn

Uniqueness. In nonlinear setting, uniqueness is false. This is clear, for example, in loading of a beam or column  where buckling can occur. Even when bc fixes $x(X)$ on the entire body, uniqueness can fail, as

This classic thought-expt due to F. John shows: consider a 2D annulus, with bdry cond $x(X) = X$ at bdry. Clearly $x(X) \equiv X$ is a soln. But we also expect equilibria in which rays $\theta = $ const wind any fixed # of times (see figure)



By contrast, in <u>linear setting</u> solutions are usually unique (the sole exception being the case of traction BC on the entire bdry, when soln is unique "up to a rigid motion").

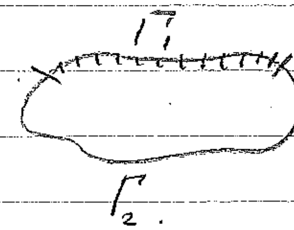
Fundamental reason: in linear elasticity, the assoc var'l principle is <u>convex</u> (in fact quadratic + linear).

Actually, a very elementary pf of uniqueness is possible, taking advantage of the quadratic character of elastic energy. As warm up, recall this simple pf of uniqueness for Laplace's eqn $\Delta u + f = 0$ in Ω, $u = u_0$ at $\partial\Omega$: if $\exists$ two solns then difference solves $\Delta w = 0$ in Ω and $w = 0$ at $\partial\Omega$. Multiply by w and integrate by parts $\Rightarrow 0 = \int w \cdot \Delta w = -\int |\nabla w|^2 \Rightarrow \nabla w = 0$.

So $w = \text{const}$; but $w=0$ at $\partial\Omega$ so $w=0$.

<u>A similar argt works for elasticity</u>: consider bvp

$$
\begin{aligned}
-\operatorname{div}(\alpha e(u)) &= f \quad \text{in } \Omega \\
u &= u_0 \quad \text{on } \Gamma_1 \\
[\alpha e(u)]\cdot n &= g \quad \text{on } \Gamma_2
\end{aligned}
$$



where $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Soln is unique.

<u>Pf</u>: need only show $f=g=0$, $u_0=0 \Rightarrow u\equiv 0$. Argue as for Laplace: with $\sigma = \alpha e(u)$,

$$0 = \int_\Omega \langle u, \operatorname{div}\sigma\rangle = -\int_\Omega \langle \nabla u, \sigma\rangle \qquad \text{by Green's thm + bc}$$

$$= -\int \langle e(u), \sigma\rangle \qquad \text{since } \sigma \text{ is symmetric}$$

$$= -\int \langle \alpha e(u), e(u)\rangle \qquad \text{defn of } \sigma$$

$$\Rightarrow \quad e(u)\equiv 0 \qquad \text{since } \alpha \text{ is pos def.}$$

Now need <u>Lemma</u>: if $e(u)\equiv 0$ in connected region Ω then u is an "inf'l rigid motion" i.e.

$$u(x) = \sum \omega_{ij} x_j + d_i$$

for some (constant) skew-symmetric ω_{ij} + some const d_i.

(Note: this is linear analogue of stmt that $F^TF \equiv I \Rightarrow x(X)$ is locally a rigid motion.)

Proof is easy in $\mathbb{R}^2$: $u_{1,1} \equiv 0$, $u_{2,2} \equiv 0 \Rightarrow u_1 = f(x_2)$

$u_2 = f(x_1)$

$u_{1,2} + u_{2,1} \equiv 0 \Rightarrow f'(x_2) + g'(x_1) = 0$

$\Rightarrow f = \omega x_2 + \text{const}$

$g = -\omega x_1 + \text{const}$

Proof in $\mathbb{R}^3$ (or $\mathbb{R}^n$, $n \geq 3$) can be done similarly, by induction on dimn. Or, here's another (less intuitive) argt: observe that

$$\partial_{jk} u_i = \partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk}$$

Therefore $e(u) \equiv 0 \Rightarrow \nabla\nabla u \equiv 0 \Rightarrow u$ is linear in x. Since $e(u) = 0$, Du is skew-symmetric.

Wrap up pf of uniqueness: we were assuming $\operatorname{div} \sigma = 0$ in Ω, $u = 0$ at Γ_1, $\sigma . n = 0$ at Γ_2. We concluded u = inf'l rigid motion. If $\Gamma_1 \neq \emptyset$ this forces $u = 0$.

What about pure traction plm? Situation

is like Neumann plm for Laplace eqn.
Recall: for $\Delta u = f$ in Ω, $\frac{\partial u}{\partial n} = g$ at $\partial\Omega$ we have a <u>consistency condn</u>: $\int_\Omega f = \int_{\partial\Omega} g$; when consistency holds, soln is unique up to a constant.

<u>Similar situation for lin elasticity</u>: traction plm

$$-\operatorname{div}\sigma = f, \quad \sigma = \mathcal{C}e(u) \quad \text{in } \Omega$$
$$\sigma \cdot n = g \quad \text{at } \partial\Omega$$

can have soln only if

$$\int_{\partial\Omega} \langle g, \hat{u}\rangle\, ds + \int_\Omega \langle f, \hat{u}\rangle\, dx = 0 \quad \text{whenever } \hat{u} \text{ is an inf'l rigid motion.}$$

If it does have a soln, that soln is unique up to addition of an inf'l rigid motion.

<u>Pf of consistency</u>: if $e(\hat{u}) \equiv 0$ then

$$\int_\Omega \langle \hat{u}, f\rangle = -\int_\Omega \langle \hat{u}, \operatorname{div}\sigma\rangle = \cancel{+\int_\Omega \langle e(\hat{u}), \sigma\rangle}^{\,0} - \int_{\partial\Omega} \langle \hat{u}, \underbrace{\sigma \cdot n}_{g}\rangle.$$

<u>Pf of uniqueness</u> : same arg't as before (except now $\Gamma_1 = \phi$ so uniqueness is only true up to inf'l rigid motion)

What about existence? Again, soln is a lot like scalar Laplace eqn, or div-form scalar eqn $\nabla\cdot(\sigma(x)\nabla u) = f$. Main techniques

① var'l principles
② Lax-Milgram lemma
$\Big\}$ very closely connected!
③ bdry integral techniques. (different)

Bdry integral methods are basically restricted to <u>constant-coefft</u> setting (won't discuss them here).

Var'l prin + Lax-Milgram are simple + general; also form basis of most numerical schemes (eg finite elements), we'll focus on former.

Again, use scalar Laplace as guide. Soln to

$$-\Delta u = f \text{ in } \Omega, \quad u = u_0 \text{ on part of } \partial\Omega \ (\Gamma_1)$$
$$\partial u/\partial n = g \text{ on rest of } \partial\Omega \ (\Gamma_2)$$

can be found using var'l prin

$$(*) \quad \min_{u=u_0 \text{ on } \Gamma_1} \int_\Omega \tfrac{1}{2}|\nabla u|^2 - fu \, dx \; - \int_{\Gamma_2} ug \, ds$$

<u>Note:</u> if $\Gamma_1 = \phi$ and data are inconsistent, then functional is unbdd below; we can drive it to $-\infty$ by taking u = suitable constant.

Existence via var'l prin uses convexity of fn'l, plus lemma that it's bdd below. Key to latter is a pair of Poincaré-type ineq's

<u>easy Poincaré ineq</u>: $\int_\Omega |u|^2 \le C \int_\Omega |\nabla u|^2$

provided $u=0$ at $\partial\Omega$

<u>hard Poincaré ineq</u>: $\int_\Omega |u-\bar{u}|^2 \le C \int_\Omega |\nabla u|^2$

where $\bar{u}$ = avg of u on Ω.

They assure us that although "energy" controls only $\int|\nabla u|^2$ directly, it also controls $\int |u|^2$ indirectly.

Situation for elasticity is just the same. Var'l prin is

$$\min_{u=u_0 \text{ on } \Gamma_1} \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle + \langle f, u\rangle \, dx - \int_{\Gamma_2} \langle u, g\rangle \, ds$$

Analogue of Poincaré ineq is <u>consequence</u> of <u>Korn's</u> $\neq$:

<u>easy version</u>: $\int_\Omega |\nabla u|^2 \le C \int_\Omega |e(u)|^2$

provided $u = 0$ at $\partial\Omega$

<u>harder version</u>: $\int_\Omega |\nabla u|^2 \le C \int_\Omega |e(u)|^2$

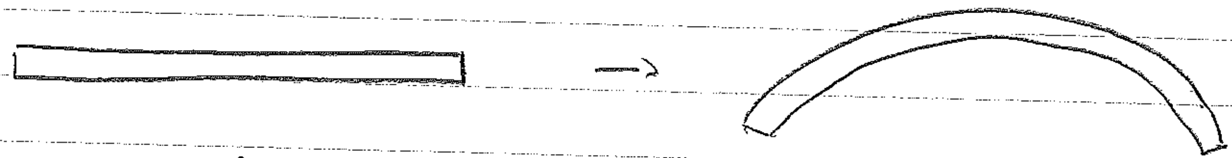
provided $\int_\Omega \nabla_i u^j$ is symmetric matrix.

They assure that although "energy" controls only $\int |e(u)|^2$ directly, it controls $\int |\nabla u|^2$ indirectly (and therefore also $\int |u|^2$ indirectly).

Same intuition on (hard) Korn ineq: it clearly implies

$$\int_\Omega |u - \hat{u}|^2 \le C \int_\Omega |e(u)|^2 \quad \text{for some inf'l rigid motion}$$

which is linear analogue of estimate (true, but much harder) that [small nonlin strain] => [close to rigid rotation]. Const depends on domain, of course, and long, thin domains $\Rightarrow$ very large constants.



locally close to a rigid motion.
But not globally!

"Easy Korn ineq" can be proved by an elementary integr by parts, or by an easy Fourier-transform-based argument. Here is the former:

for $u \in C_0^\infty(\Omega)$ (with Ω bdd),

$$\int_\Omega |e(u)|^2 = \int_\Omega \sum \left(\frac{\nabla_i u^j + \nabla_j u^i}{2} \right)^2$$

$$= \int_\Omega \frac{1}{2} |\nabla u|^2 + \int_\Omega \frac{1}{2} \sum \nabla_j u^i \nabla_i u^j$$

But since $u = 0$ near $\partial\Omega$,

$$\int_\Omega \nabla_j u^i \nabla_i u^j = \int_\Omega \nabla_i u^i \nabla_j u^j$$

So

$$\int_\Omega |e(u)|^2 = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} \int |\operatorname{div} u|^2$$

Thus when $u=0$ at $\partial\Omega$ Korn's ineq with const C says

$$\int_\Omega |\nabla u|^2 \le C\left(\int_\Omega \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}(\operatorname{div} u)^2\right)$$

Evidently true with $C=2$.

"Hard Korn ineq" has interesting history:

- 1st "proof" by Korn, about 1910
- Friedrichs wrote a paper abt 1947 pointing out importance of this inequality, giving a new (still not simple) proof, + 1st "modern" proof of existence of solns (by Hilbert-space methods)
- Many proofs since Friedrichs. Some are very efficient but not so elementary (see eg Duvaut + Lions book). Also much study of other "coerciveness inequalities": when does control of selected combinations of $\partial_j u^i$ yield control of all derivatives individually (KT Smith, Aronszajn, others — 60's + 70's,

using pseudodifferential operators).

- A really simple, elementary proof was finally given by Oleinik & Kondratiev about 1989 (CRAS Paris Ser I, 1989, 483-487; also Rend. Mat. Appl. (7) 10, 1990, no 3, 641-666). I'll distribute a handout on this.

I promised a discussion of existence via variational principle. Actually let's do a little less (I won't actually prove existence — though those who know the var'l pf of existence for Laplace's eqn will see what to do) and a little more (I'll discuss convergence of a typical finite element method calculation). Focus on:

linear elasticity in Ω (no body load)
$u = 0$ on part of $\partial\Omega$ (call it Γ_1)
$\sigma \cdot n = g$ on rest of $\partial\Omega$ (call it Γ_2).

for which var'l prin is

$$\min_{u=0 \text{ on } \Gamma_1} \int_\Omega \frac{1}{2}\langle \alpha e(u), e(u)\rangle - \int_{\Gamma_2} \langle u, g\rangle$$

Typical <u>numerical method</u>: minimize exactly the same functional in a finite dim'l space of functions V (chosen so that $u|_{\Gamma_1}=0$ whenever $u \in V$). For example: V could consist of fns that are piecewise linear + cont's on a fixed triangulation of Ω.

Let $E[u] = \int_\Omega \frac{1}{2}\langle \alpha e(u), e(u)\rangle - \int_{\Gamma_2} \langle u, g\rangle$

+ let u_* = minimizer (i.e. actual soln of elasticity plm). Since E is quadratic + linear, Taylor exp. around u_* keeping only quadratic terms is exact. So for any v,

$$E(v) = E(u_*) + \underbrace{\boxed{\int_\Omega \langle \alpha e(u_*), e(v)-e(u_*)\rangle\,dx - \int_{\Gamma_2}\langle v-u_*, g\rangle\,ds}}_{1^{st}\text{ varn}}$$

$$+ \underbrace{\frac{1}{2}\int_\Omega \langle \alpha(e(v)-e(u_*)), e(v)-e(u_*)\rangle}_{2^{d}\text{ varn}}$$

The eqns of elasticity (the EL eqns for E at u_*) assure that boxed terms (the 1^{st} varn) vanish, so

$$(*) \quad \frac{1}{2} \int_\Omega \langle \alpha\, e(v-u_*),\, e(v-u_*) \rangle = E[v] - E[u_*]$$

If u_* can be approx well in the subspace V then RHS will be small at the best $v \in V$

<u>Claim</u>: if RHS is small, then v is close to u_* in H^1. This follows immediately from $(*)$ using positivity of α and the Korn ineq

$$(**) \quad \int_\Omega |\nabla w|^2 \leq C \int_\Omega |e(w)|^2 \quad \text{if } w = 0 \text{ on } \Gamma_1.$$

Explanation of $(**)$: well, by the "hard" Korn ineq

$$\int_\Omega |\nabla w - \gamma|^2 \leq C \int_\Omega |e(w)|^2$$

for some skew-symmetric, constant matrix γ. A Poincaré-type ineq then gives

$$\int_\Omega |w - (\gamma \cdot x + d)|^2 \leq C \int_\Omega |e(w)|^2$$

and a standard "trace" theorem for $H^1(\Omega)$ gives

$$\int_{\Gamma_1} |w - (\gamma \cdot x + d)|^2 ds \leq C \int_\Omega |e(w)|^2.$$

But $w = 0$ on Γ_1, and the map

$$(\gamma, d) \rightarrow \left(\int_{\Gamma_1} |\gamma \cdot x + d|^2 ds \right)^{1/2}$$

is a norm on the finite-dimensional space of all γ's + d's. So $\|\gamma\| + |d| \leq C' \int_\Omega |e(w)|^2$. Thus finally

$$\int_\Omega |\nabla w|^2 \leq C'' \int_\Omega |e(w)|^2$$

by triangle inequality.