Mechanics - Lecture 6, 3/6/2019

[3/6 class will begin with end of "Lecture 5 notes" — pp 5.5 - 5.9]

We've been doing "fully nonlinear" elasticity. Now let's pass to linear elasticity. This entails two separate linearizations:

1^{st} geometrical linearization ("small strain theory") — we suppose

$$\chi_\delta(X) = X + \delta u(X)$$

with δ small. Here u is the "infinitesimal elastic displacement". Evidently

$$F = I + \delta \cdot Du$$

$$\Rightarrow (F^T F)^{1/2} = I + \frac{\delta}{2}(Du + Du^T) + \mathcal{O}(\delta^2)$$

so to leading order, $(F^T F)^{1/2} - I$ is δ times the linear elastic strain

$$e(u) = \frac{1}{2}(Du + Du^T)$$

2nd physical linearization ("linear stress-strain law")

Using hyperelasticity as a starting pt (spatially homogeneous, for simplicity) with $W(I) = 0$ + $W \geq 0$ (so ref state is a local min of elastic energy), we have

$$W(F) = \delta^2 \langle \alpha e(u), e(u) \rangle + O(\delta^3)$$

where

$$\langle \alpha e, e \rangle = \sum{}' \alpha_{ijkl} \, e_{ij} \, e_{kl}$$

is a pos def (symmetric) quadratic form on symmetric tensors.

Dropping terms of higher order in δ (and rescaling by δ^2) we expect a var'l prin of the form $\delta E = 0$ where

$$E = \int_\Omega \langle \alpha e(u), e(u) \rangle \, dx + \begin{bmatrix} \text{terms assoc} \\ \text{to loads} \end{bmatrix}$$

so equil eqn is

$$\mathrm{div}\left(\alpha e(u) \right) + f = 0$$

and we recognize

$$\sigma = \alpha e(u)$$

as the stress tensor in the linearly elastic setting.

Notes: a) σ is symmetric

b) we lose the ability to distinguish between ref + deformed coords when we linearize.

(Put differently: The Cauchy + Piola-Kirchhoff stresses agree to 1^{st} order in δ; that's why σ is symmetric.)

Summary: in linear elasticity, basic unknown is "displacement" $u(x)$. Assoc lin strain is $e(u) = \frac{1}{2}(Du + Du^T)$. Stress is $\sigma = \alpha e(u)$. Eqns of equil are

$$\operatorname{div} \sigma + f = 0, \quad \sigma = \alpha e(u)$$

augmented by suitable bc, for example

① $u = u_0$ at $\partial\Omega$ ("displacement bc)

or ② $\sigma \cdot n = g$ at $\partial\Omega$ ("traction bc") or ③ $u \cdot n = 0$, $(\sigma \cdot n)_{\tan} = 0$ at $\partial\Omega$ ("lubricated bc")

We discussed isotropy in nonlinear setting. Analogue for linear elasticity is

Hooke's law α is isotropic $\iff$ $\alpha(R^T e R) = R^T(\alpha \cdot e)R$ for any e (symmetric) and any R (rotation)

(More generally: R is a symmetry of the material if $\alpha(R^T e R) = R^T(\alpha \cdot e)R$ for any e. Intuition: rotating material $\iff$ keep material fixed but rotate coordinates.)

Lemma: The general isotropic Hooke's law can be expressed as

$$\sigma_{ij} = 2\mu\, e_{ij} + \lambda(\mathrm{tr}\, e)\, \delta_{ij}$$

where λ + μ are constants ("Lamé moduli").

Expln (not quite a proof): Hooke's law isotropic $\iff$ energy $\langle \alpha e, e \rangle$ is a quadratic, isotropic function of symm tensor e. Fact of linear algebra: all such fns are obtained by starting from

$$e_{ij}\, e_{kl}$$

and "contracting indices" in pairs. There are two distinct ways to do this:

a) $\sum_{i,k}' e_{ii}\, e_{kk} = (\mathrm{tr}\, e)^2$

b) $\sum_{i,j}' e_{ij}\, e_{ij} = \mathrm{tr}(e^2) = |e|^2$

So

$$\langle \alpha e, e\rangle = 2\mu |e|^2 + \lambda (\mathrm{tr}\, e)^2,$$

for some $\lambda + \mu$.

There are other common repns of an isotropic Hooke's law, and good reasons to use them:

1st in terms of Young's modulus E + Poisson's ratio ν

$$\sigma_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu} (\mathrm{tr}\, e)\, \delta_{ij}\right)$$

2nd in terms of bulk modulus K and shear modulus μ,

$$\sigma_{ij} = 3K\left(\tfrac{1}{3}(\mathrm{tr}\,e)\,\delta_{ij}\right) + 2\mu\left(e_{ij} - \tfrac{1}{3}(\mathrm{tr}\,e)\,\delta_{ij}\right)$$

Positivity requires

$\mu > 0$	$E > 0$	$\mu > 0$
$3\lambda + 2\mu > 0$	$-1 < \nu < 1/2$	$K > 0$
for Lamé moduli	for Young's modulus + Poisson's ratio	for bulk + shear moduli

Of course one set of pars determines the others, eg

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

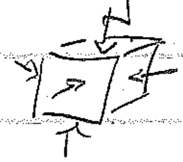
$$\lambda = K - \tfrac{2}{3}\mu$$

Meaning of these parameters:

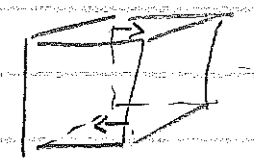
- Bulk + shear moduli describe action of α on multiples of identity + trace-free matrices ("pure shears") separately

$$\sigma_{ij} = 3K\underbrace{\left(\tfrac{1}{3}\,\mathrm{tr}\,e\,\delta_{ij}\right)}_{\text{proj'n of } e \text{ onto mult of } I} + 2\mu\left(\underbrace{e_{ij} - \tfrac{1}{3}(\mathrm{tr}\,e)\,\delta_{ij}}_{\text{proj'n of } e \text{ orthog to } I}\right)$$

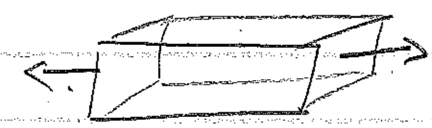
<u>Note:</u> bulk modulus measures vol change due to hydrostatic pressure



shear modulus gives response to any pure shear, eg $\sigma_{12} = T$, $\sigma_{ij} = 0$ otherwise
$\Rightarrow e_{12} = \frac{1}{2\mu}\sigma_{12}$



- Poisson's ratio + Young's modulus describe behavior under uniaxial tension
($\sigma_{11} = T$, other $\sigma_{ij} = 0$).



$\Rightarrow e_{11} = \frac{1}{E} T$

$e_{22} = e_{33} = \frac{-\nu}{E} T$

Most materials have $\nu > 0$, so uniaxial tension produces contraction in the orthog vars. Cork <u>has $\nu \approx 0$</u>, which is why it is used for closing wine bottles.

There are lots of scalar or 2D reductions of elastostatics that help generate intuition; see eg Howell, Kozyreff, Ockendon on

- antiplane strain (Sec 2.3)
- torsion (Sec 2.4, 2.5)

} reduction to scalar 2nd order pde

- plane strain (Sec 2.6) – a 2nd order system but equiv to a 4th order scalar pde pbm.

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var'l principles: soln of elastostatics with Dir bc (displacement fixed at $\partial\Omega$) achieves

$$\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle - \langle u, f\rangle \, dx$$

Traction bc (bc $\sigma\cdot n = F$ at $\partial\Omega$) – we have similar var'l principle

$$\min_u \left\{ \int_\Omega \tfrac{1}{2}\langle \alpha e(u), e(u)\rangle - \langle u, f\rangle \, dx - \int_{\partial\Omega} \langle u, F\rangle \right\}$$

(we'll discuss these + their implications next week). These are the elastic analogues of the well-known var'l prin's for Laplace's eqn

$-\Delta u = f$ in Ω, $u = \varphi$ at $\partial\Omega$ $\Longleftrightarrow$ u achieves min of

$$\int_\Omega \tfrac{1}{2}|\nabla w|^2 - wf \, dx$$

among all fns st $w=\varphi$ at $\partial\Omega$

$-\Delta u = f$ in Ω, $\frac{\partial u}{\partial n} = F$ at $\partial\Omega$ $\Longleftrightarrow$ u achieves min of

$$\int_\Omega \tfrac{1}{2}|\nabla w|^2 - wf \, dx - \int_{\partial\Omega} wF \, dA$$

among all fns w (no bc) on Ω

Some thing's we'll need to sort out:

- for traction bc there are consistency conds on $f + F$, namely
$$\int_\Omega \langle u, f\rangle \, dx + \int_{\partial\Omega} \langle u, F\rangle \, dA = 0$$
whenever $e(u) \equiv 0$

- $e(u) \equiv 0$ iff $u_i(x) = \sum_j c_{ij} x_j + d_i$ with $c_{ij} + d_i$ constant + c_{ij} skew-symmetric (such u is an "infinitesimal rigid motion")

- Korn's inequality:

easy version: $\int_\Omega |\nabla u|^2 \le C_1 \int_\Omega |e(u)|^2$ if $u|_{\partial\Omega} = 0$

harder version: $\int_\Omega |\nabla u|^2 \le C_2 \int_\Omega |e(u)|^2$ if $\int_\Omega \nabla u$ is symmetric

note key consequence (combining these with a Poincaré-type inequality):

- $\int_\Omega |u|^2 \le C_1' \int_\Omega |e(u)|^2$ if $u|_{\partial\Omega} = 0$.

- $\int_\Omega |u-\hat{u}|^2 \le C_2' \int_\Omega |e(u)|^2$ for some infinitesimal rigid motion $\hat{u}$

Constant C_2 depends on Ω, as one easily sees by considering long, thin domains, where C_2 must be large since $\exists$ deformations with small strain that are not close to rigid motions