Mechanics – Lecture 5 – 2/27/2019.

[We'll start by finishing the Lecture 4 notes.]

We turn now to constitutive laws for nonlinear elasticity.

Two viewpoints are possible:

(a) "Cauchy elasticity": specify Cauchy stress as function of reference position + def gradient

$$\Sigma = \hat{\Sigma}(X, F)$$

(b) "Hyperelasticity": specify $P_{i\alpha} = \dfrac{\partial W(X,F)}{\partial F_{i\alpha}}$

where W = "elastic energy density"

Often a body might be homogeneous in its ref configuration; this means $\hat{\Sigma} = \hat{\Sigma}(F)$ or $W = W(F)$ (indep of position X).

We always require the structural condition of "frame indifference"

a) for Cauchy elasticity: $\hat{\Sigma}(RF) = R\,\hat{\Sigma}(F)R^T$ for all orientation-preserving rotations R

b) for hyperelasticity: $W(F) = W(RF)$ for all orientation preserving rotations R.

Interpretation:

a) observer in rotated coord system sees same basic stress-strain law

b) rotations do no work.

HW3 will ask you to show the equiv of these relations, for hyperelasticity. But let me do here a key consistency check:

<u>Claim</u>: if $P_{i\alpha} = \frac{\partial W}{\partial F_{i\alpha}}$ where W is frame indifferent, then the assoc Cauchy stress Σ is symmetric

<u>Pf</u>: Since $P = J\,\Sigma\,(F^{-1})^T$ $\quad (J = \det F)$ we have

$$\Sigma = J^{-1} P F^T$$

so we must show that $PF^T = \sum P_{i\alpha} F_{j\alpha}$ is symmetric. For any skew-symmetric matrix Ω_{ij} there's a rotation-valued curve $R(t)$ s.t. $R(0) = I$ and $\dot{R}(0) = \Omega$. Frame indifference gives

$$0 = \frac{d}{dt}\Big|_{t=0} W(R(t)F) = \sum_{i,\alpha} P_{i\alpha}(F)\,(\dot{R}F)_{i\alpha}$$

$$= \sum_{i,\alpha,j} P_{i\alpha}(F)\,\Omega_{ij}\,F_{j\alpha}$$

So $P(F)F^T$ is orthogonal to the space of skew-symmetric matrices. So it is symmetric.

Personally, I prefer hyperelasticity (there seems to be little incentive to look beyond this framework to the more general setting of Cauchy elasticity).

Many materials are <u>isotropic</u>. In hyperelasticity

isotropy $\iff W(F) = W(FR)$ for any orientation preserving rotn R

$\iff W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ where $\{\lambda_i\}$ are the principal stretches (eigenvalues of $(F^TF)^{1/2}$) and φ is a <u>symmetric</u> function of it's arguments.

Corresp. assertion for Cauchy elasticity:

isotropy $\iff \hat{\Sigma}(FR) = \hat{\Sigma}(R)$ for all R.

For strings it was natural to ask that $\nu \to N(\nu)$ be monotone increasing. Similarly, for 3D elasticity it is natural to impose some structural conditions, eg

a) eqns of elastostatics are elliptic (and eqns of elastodynamics are hyperbolic) or, more globally,

b) var'l prin of elastostatics achieves its minimum.

Discn of (b) would take us too far afield (keyword: "W should be quasiconvex."). Essence of (a) is that $F \to W(F)$ should be (strictly) "rank one convex"

$$\sum \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} \xi_i \xi_j \eta_\alpha \eta_\beta \geq C |\xi|^2 |\eta|^2$$

for all $\xi, \eta \in \mathbb{R}^3$. Intuition about why it matters: for a const coefft linear pde system

$$\sum_{\alpha, j, \beta} \frac{\partial}{\partial X_\alpha} \left(A_{i\alpha j\beta} \frac{\partial x_j}{\partial X_\beta} \right) = f_i \quad \text{in } \mathbb{R}^n$$

we can try to solve by Fourier transform

$$- \sum_{\alpha, \beta, j} k_\alpha k_\beta A_{i\alpha j\beta} \hat{x}_j(k) = \hat{f}_i(k)$$

Cond (*) assures that the matrix $\sum_{\alpha, \beta} k_\alpha k_\beta A_{i\alpha j\beta}$ (which has to be inverted) is pos definite.

Rank-one convexity is weaker than convexity. Why not just assume elastic energy is a convex fn of F? Ans: it's not compatible with frame indifference + condn that W(F) be min when F is a rotn

$$W \text{ is min at } F \in SO(3) \quad (\text{only})$$
$$+ \; SO(3) \text{ is not convex}$$
$$\Rightarrow W \text{ cannot be convex}$$

But convexity is still a convenient tool for identifying suitable energy fns. For example

$$(*) \qquad \text{if } W(F) = \text{convex fn of } F$$
$$\qquad\qquad + \text{convex fn of } \det F$$

Then W is rank-one convex (indeed, "quasi-convex"), and such structure <u>is</u> compatible with the desired behavior. (How to find a frame-indifferent convex fn of F? Use this result, proved eg in Ciarlet's book as Thm 4.9-1: if $W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$ where λ_j are eigs of $(F^TF)^{1/2}$ + φ is symmetric, convex, + nondecreasing in each var, then W is a convex fn of F.)

A simple (extreme) case of (*) is the <u>incompressible</u> <u>neoHookean</u> material

$$W(F) = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad \text{restr by } \det F = \lambda_1\lambda_2\lambda_3 = 1$$

(note: $\lambda_1^2+\lambda_2^2+\lambda_3^2 = \mathrm{tr}(F^TF) = \sum F_{i\alpha}^2$ is obviously convex!).

There's much more to say here - the main tool for getting rank-one convex W is to use "polyconvexity" - but this would take us too far afield. See Ch 4 of Ciarlet for that.

We briefly touched on incompressibility just now. In fact rubber is nearly incompressible and the incompr neoHookean law is widely used as a model constitutive law. What does eqn of elastostatics look like in this setting? Short ans: pde includes an unknown fn - the <u>pressure</u> - as a Lagrange multiplier for constraint of incompressibility; assoc constit law is therefore

$$\Sigma = -p\,I + \Sigma^*(F)$$

where Σ^* is given by an elastic energy by rules discussed earlier (but restricted to $\det F = 1$) & p is determined by balance of forces.

<u>Explain this</u>: starting from constrained var'l prin

$$0 = \delta_{\det(\partial x/\partial X)=1} \int_\Omega W(\partial x/\partial X)\, dX$$

$\Rightarrow$ by method of Lagr mult, EL eqn is formally

$$\delta \int_\Omega W(\partial x/\partial X) + \gamma(X)\left[\det\left(\frac{\partial x}{\partial X}\right) - 1\right] dX = 0$$

for some (unknown) $\gamma(X)$. Assoc pde is

$$\sum_\alpha \frac{\partial}{\partial X_\alpha}\left(\frac{\partial W}{\partial F_{i\alpha}} + \gamma(X)\frac{\partial(\det F)}{\partial F_{i\alpha}}\right) = 0.$$

Our task is to show that Cauchy stress assoc to Piola-Kirchhoff stress $\gamma(X)\dfrac{\partial \det F}{\partial F_{i\alpha}}$ has the form $-p(x)I$. (In fact we'll show $p = -\gamma$). Key is Cramer's rule, which says $\dfrac{\partial(\det F)}{\partial F_{i\alpha}}$ = matrix of minors = $J(F^{-1})^T$

Recalling that $P = J\tau (F^{-1})^T$ i.e. $\tau = J^{-1} P F^T$ we get

$$P = \gamma \frac{\partial \det F}{\partial F_{i\alpha}} \Rightarrow \tau = J^{-1}\left(\gamma J (F^{-1})^T\right) F^T$$
$$= \gamma I$$

as asserted.

Returning to general (compressible) case, here's another viewpt that is sometimes useful. Recall that for isotropic elasticity,

$$W(F) = \varphi(\lambda_1, \lambda_2, \lambda_3)$$

where φ is a symmetric fn of 3 vars + $\{\lambda_j\}$ are eigs of $(F^TF)^{1/2}$. Any such fn has the alternative repn

$$(*) \quad W(F) = \psi(I, II, III)$$

where I, II, III are the elementary symmetric fns of the eigenvalues of $F^TF = C$, ie.

$$I = \operatorname{tr} C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$II = \tfrac{1}{2}\left[(\operatorname{tr} C)^2 - \operatorname{tr}(C^2)\right] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$$

$$III = \det C = \lambda_1^2 \lambda_2^2 \lambda_3^2$$

Advantage of this repn: Piola-Kirchhoff stress involves a polynomial expression in derivs of Ψ and components of F.

There is an analogous framework for constitutive modeling in Cauchy elasticity:

$$(**) \quad \Sigma(F) = \varphi_0 I + \varphi_1 B + \varphi_2 B^2$$

where $B = FF^T$ and φ_i are suitable fns of I, II, III. (Note: $B = FF^T$ & $C = F^T F$ have the same eigenvalues, though their eigenvectors are generally different.)
HW3 will ask you to show $(*) \Rightarrow (**)$.

HW3 will also have pbms exploring

- how expts can be used to identify the constitutive law
- how nonlinearities of elasticity explain observable effects, eg rln btwn pressure & radius when blowing up a balloon.

(Lecture 6 will turn to linear elasticity.)