Mechanics – Lecture 4, 2/20/2019

Continuing with a bit more about deformation + strain:

Question 4: What can we say about deformations that do little stretching? Much harder question! Studied by F. John in the 1960's then revisited by Friesecke, James, + Müller about 2002. They showed:

$$(*) \quad \inf_{R \in SO(3)} \| Du - R \|^2_{L^2(\Omega)} \leq C \int_\Omega \text{dist}^2(Du, SO(3))$$

(a "nonlinear Korn ≠") + they used it to provide rigorous justifications of various plate + shell theories.

[The linear-elastic analogue of * is a basic estimate in linear elasticity; we'll return to that in due course.]

Question 5: What about elastic membranes? Their deformations are maps $\mathbb{R}^2 \to \mathbb{R}^3$. Polar decomp (really, SVD) gives $F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha} = R \cdot U$ where $U = (F^T F)^{1/2}$ is now 2x2, and R is an isometric immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$.

Which defs do no stretching or shrinking?

Image is then a 2D surface in $\mathbb{R}^3$ isometric to $\mathbb{R}^2$. Nec + suff't condn (assuming C^2) is that Gaussian curvature vanish. Fact of diff'l geometry: such surfaces are "developable" (directions with prin curvature zero line up, forming straight lines that stay in the surface).

OK, now let's turn to stress (analysis of forces; "statics"; Cauchy's thm is closely linked to the fact that the natural objects to integrate over 2D surfaces are 2-forms)

We consider 2 types of forces:

- body force $\vec{f}(x)$ per unit deformed volume, acting at (deformed) position x
- surface force $\vec{T}(x; \vec{n})$ per unit deformed area, acting on plane $\perp$ unit vector $\vec{n}$ at (deformed) position x. (Sign convention: $\vec{T} = -p\vec{n}$ with $p > 0$ for hydrostatic pressure

Intuition: a spatial volume is acted upon by body force (eg gravity) and also a surface force (applied by rest of body).

Note convention: $\vec{f}$ = force/unit deformed vol. Assoc force/unit ref vol is $f \cdot \det\left(\frac{\partial x}{\partial X}\right)$. (In lecture 1 we used f for force/unit ref length; that was a different convention.)

Clearly $\vec{T}(\vec{n}) = -\vec{T}(-\vec{n})$ ("law of equal reaction")

Cauchy's Thm:

a) $\vec{T}(x, \vec{n})$ is linear in $\vec{n}$, i.e.

$$T_i(x, \vec{n}) = \sum_{j=1}^{3} \tau_{ij}(x)\, n_j$$

b) The matrix $\tau_{ij}(x)$ is symmetric, i.e.

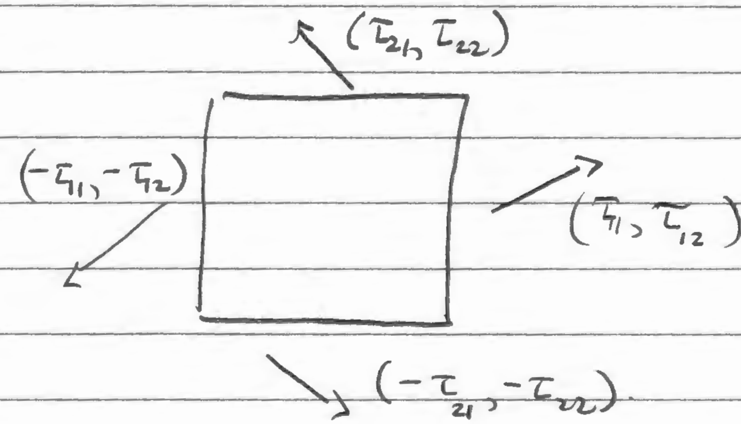
$$\tau_{ij} = \tau_{ji} \qquad \text{for all } i \neq j$$

We call τ the Cauchy stress tensor. (In dynamics, $\vec{T} = \vec{T}(x, t; \vec{n})$; same result holds, with $\tau = \tau_{ij}(x, t)$.)

Intuition behind (a): it says essentially that stress is a 2-form. (Proof resembles discn in H. Whitney's book "Geometric Integration Theory" – 1st few pages only! – why 2-forms are the natural objects to integrate over surfaces.)

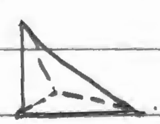
Intuition behind (b): if it weren't so then a small cube of material would experience a net torque. Visualization in 2D:

if $\tau_{12} \neq \tau_{21}$ forces on faces would be as shown [figure has $\tau_{21} < 0$, $\tau_{12} > 0$]



<u>Pf of Cauchy's Thm</u>, assuming everything is in static equilibrium (the dynamic case is only slightly different)

<u>Pf of (a)</u>: Fix x_0 + $\vec{n}$. Consider tetrahedron (in deformed coords) V with corner x_0 + 3 faces $\perp$ axes (call them S_1, S_2, S_3) + 4th face $\perp \vec{n}$ (call it S_0). Figure, for $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$:



Geometry $\Rightarrow$ $|S_i| = n_i |S_0|$

Outward normal to S_i is $-e_i$

Balance of forces =>

$$0 = \int_{S_0} T(x,\vec{n})\, dA + \sum_i{}' \int_{S_i} T(x,-e_i)\, dA + \int_V f\, d\mathrm{vol}$$

where dA = surface area. Assuming a little regularity, divide by $|S_0|$ + take small-vol limit to get

$$T(x,\vec{n}) + \sum n_i T(x,-e_i) = 0$$

(Vol integral scales like vol not area, so it $\to 0$ in limit if f is bdd). Thus:

$$T(x,\vec{n}) = \sum_j \tau_{ij}(x) n_j \quad \text{, with } \tau_{ij} = T_i(x, e_j).$$

[Proof in dynamic setting is almost the same; sole difference: there's an additional body force assoc acceleration x_{tt}.]

We also see equil pde at this stage:

$$(*) \qquad \sum_{j=1}^{3}{}' \frac{\partial \tau_{ij}}{\partial x_j} + f_i = 0$$

since total force on any volume must vanish.

Pf of (b): equil eqn came from balance of forces; symmetry comes from balance of torques: in static setting

$$\int_V x \wedge f \, dvol + \int_{\partial V} x \wedge (\tau \cdot n) \, dA = 0$$

where $a \wedge b$ is vector cross-product of $a, b \in \mathbb{R}^3$:

$$(a \wedge b)_i = \sum_{j,k} \varepsilon_{ijk} a_j b_k = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

PDE version of balance of torques is

$$(**) \quad \sum_{j,k} \varepsilon_{ijk} x_j f_k + \sum_{j,k,l} \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j \tau_{kl} \right) = 0$$

Simplify using bal of forces (*) to get

$$\sum_{jkl} \varepsilon_{ijk} \delta_{jl} \tau_{kl} = 0 \qquad \text{for each } i$$

ie

$$\sum_{jk} \varepsilon_{ijk} \tau_{kj} = 0 \qquad \text{for each } i$$

whence τ is symmetric.

Less coordinate-bound version of this calcn:

$$\int_{\partial V} x_i T_j(n) - x_j T_i(n) = \int_{\partial V} \sum_k (x_i \tau_{jk} - x_j \tau_{ik}) n_k$$

$$= \int_V \sum_k \frac{\partial}{\partial x_k} (x_i \tau_{jk} - x_j \tau_{ik})$$

$$= \int_V \tau_{ji} + x_i (\mathrm{div}\, \tau)_j$$

$$- \tau_{ij} - x_j (\mathrm{div}\, \tau)_i$$

$$= \int_V (\tau_{ji} - \tau_{ij}) + (x_j f_i - x_i f_j)$$

Now, balance of torques says

$$0 = \int_{\partial V} (x_i T_j - x_j T_i) + \int_V x_i f_j - x_j f_i$$

for all $i \neq j$. Combined with calcn above we have

$$\int_V (\tau_{ij} - \tau_{ji}) \, dx = 0$$

True for all $V \Rightarrow \tau_{ij} = \tau_{ji}$.

Done with stress + strain. Also done with balance laws (at least for statics). What's left? We must discuss

- constitutive laws (the relation b'twn stress + strain)
- "reference" vs "deformed" (Lagrangian vs Eulerian) coordinates
- examples

These notes do a simple 1st pass at constitutive laws, then focus on the reln b'twn Lagrangian + Eulerian coordinates. (We'll do more on constitutive laws in Lecture 5; you'll do some examples in HW 3.)

Constitutive laws + "ref" vs "deformed" coords are intertwined, since easiest approach to constitutive laws is via var'l principles (which works best in ref. coords) but measurements + most practical loadings (eg pressure) come to us as force / unit deformed area.

<u>1st pass at constitutive law</u>:

elastostatic equilibrium $\Leftrightarrow$ $\delta E = 0$, where

$$E = \int_\Omega W(\partial x/\partial X)\, dX + \int_\Omega V(x(X))\, dX$$

with Ω = reference config^n + $W(F)$ a fn of matrices ("elastic energy density") satisfying certain structural conditions (to be discussed later). Assoc pde is

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial X_\alpha}\left[\frac{\partial W}{\partial F_{i\alpha}}(\partial x/\partial X)\right] - \frac{\partial V}{\partial x_i}(x(X)) = 0,$$

which expresses balance of forces in <u>ref</u> <u>coords</u>

$\frac{\partial W}{\partial F_{i\alpha}}(\partial x/\partial X)$ = force in dir^n i per unit <u>ref</u> area, acting on surface $\perp$ α^{th} coord vector in ref coordinates

= defn "1st Piola-Kirchhoff stress tensor" $P_{i\alpha}$

$-\frac{\partial V}{\partial x_i}(x(X))$ = force per unit <u>ref</u> vol acting at location $x(X)$

This viewpt is convenient because we're solving a pde in a <u>known</u> domain Ω.

<u>Relation btwn ref + deformed quantities</u>: recall that

τ_{ij} = force in dir^n i per unit <u>deformed</u> area, acting on surface $\perp$ j^{th} coord vector in <u>deformed</u> coords and that τ is symmetric ($P_{i\alpha}$ is <u>not</u> symmetric!)

A common bc cond is "constant pressure p_0" which means

$$\tau \cdot \vec{n} = -p_0 \vec{n} \quad \text{at } \partial x(\Omega),$$

where $\vec{n}$ = unit normal to $\partial x(\Omega)$. Simple to say in Eulerian coords, messy to write in Lagr coords.

"Traction free" bc is easy in both contexts

$$\tau \cdot \vec{n} = 0 \text{ at } \partial x(\Omega), \quad \vec{n} = \text{unit normal to } \partial x(\Omega)$$

$$\Updownarrow$$

$$P \cdot N = 0 \text{ at } \partial\Omega, \quad N = \text{unit normal to } \partial\Omega$$

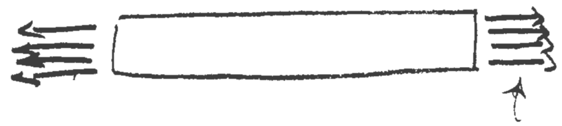
but specified (nonzero) traction is more subtle; the statements

$$P \cdot N = f \quad (\text{"dead load" } f)$$

vs

$$\tau \cdot n = f \quad (\text{"live load" } f)$$

are <u>different</u>. (Dead loads are hard to apply in practise, since load must maintain its direction + magnitude / unit ref area regardless of deformation. One scheme: use long springs, eg for uniaxial tension



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Key to translation btwn Eulerian + Lagrangian:

$$\sum_{j=0}^{3} \frac{\partial}{\partial x_j}(T_{ij}) + f_i = 0$$ (with f = force per unit deformed vol)

$$\Updownarrow$$

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial X_\alpha}(P_{i\alpha}) + f_i^R = 0$$ with $f^R = f \det\left(\frac{\partial x}{\partial X}\right)$ = force / unit vol in ref coords

<u>Claim</u> $P_{i\alpha} = \det\left(\frac{\partial x}{\partial X}\right) \sum_k T_{ik} \frac{\partial X_\alpha}{\partial x_k}$

(In matrix notation: $P = J\,T(F^{-1})^T$ where $F_{i\alpha} = \partial x_i / \partial X_\alpha$

and $J = \det F$.)

<u>Pf of Claim</u>: $\sum_j \frac{\partial}{\partial x_j}(T_{ij}) + f_i = 0 \iff \int -\sum_j T_{ij} \frac{\partial \varphi}{\partial x_j} + \sum f_i \varphi \, dx = 0$

for all cptly spt'd φ (here i is fixed)

Change to X vars:

$$\int \left[-\sum_{j,\alpha}' \tau_{ij} \frac{\partial \varphi}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial x_j} + f_i \varphi \right] \left(\det \frac{\partial x}{\partial X} \right) dX = 0 .$$

$$\sum_\alpha' \frac{\partial}{\partial X_\alpha} P_{i\alpha} + f_i^R = 0 \quad \text{with} \quad P_{i\alpha} = \sum_j' \tau_{ij} \frac{\partial X_\alpha}{\partial x_j} \det \frac{\partial x}{\partial X} .$$

[Warning: as noted earlier, $P_{i\alpha}$ is __not__ symmetric.]

Another place where Eulerian vs Lagr. is relevant is the comparison btwn elastodynamics + fluid dynamics.

__Eqns of elastodynamics, in Lagr vars:__

$$\sum_\alpha \frac{\partial}{\partial X_\alpha} (P_{i\alpha}) + m(X) g_i = m(X) \ddot{x}_i$$

where
$m(X)$ = mass per unit ref vol (density)
$m(X) g$ = body force per unit ref vol.
$\ddot{x}$ = 2nd deriv in t, holding ref. posn fixed.

To recognize cons of mass + momentum as consequences, we should write this in Eulerian coords. It's impt to distinguish

$\frac{Df}{Dt}$ = deriv of f wrt t, holding X fixed

$\frac{\partial f}{\partial t}$ = deriv of f wrt t, holding x fixed

By chain rule:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \sum_i v_i \frac{\partial f}{\partial x_i} \qquad \text{where } v_i = \frac{Dx_i}{Dt}$$

(we wrote $\dot{x}_i$ for v_i on page 4.5)

<u>Claim</u>: Eqns of elastodynamics in Eulerian coords are

$$\frac{\partial \rho}{\partial t} + \sum_j \frac{\partial}{\partial x_j}(\rho v_j) = 0 \qquad [\text{cons of mass}]$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v \cdot \nabla_x v^i\right) = \sum_j \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i \qquad \left[\begin{array}{l}\text{cons of}\\ \text{momentum}\end{array}\right]$$

where

$$\rho(x,t) = m(X(x,t)) \det\left(\frac{\partial X}{\partial x}\right) = \text{mass per unit deformed vol}$$

$$v(x,t) = \text{velocity (at deformed position } x\text{, time } t)$$

<u>Pf of 1st eqn</u>:
$$0 = \frac{d}{dt}\int_B m \, dX \qquad \text{for any region } B \subset \Omega$$

$$= \frac{d}{dt}\int_{x(B,t)} \rho \, dx$$

$$= \int_{x(B,t)} \rho_t \, dx + \int_{\partial x(B,t)} \rho \; v \cdot n$$

$$= \int_{x(B,t)} \rho_t + \mathrm{div}_x(\rho v) \; dx$$

True for all $B \Rightarrow \rho_t + \mathrm{div}_x(\rho v) = 0$.

Note that this eqn is "purely kinematic" (ie it makes no use of eqns of motion, forces, etc)

<u>Pf of 2nd eqn</u>: recall that $\mathrm{div}_X P = J \, \mathrm{div}_x T$, with $J = \det(\partial x/\partial X)$. So

$$m\ddot{x} = \mathrm{div}_X P + mg \iff J^{-1} m \ddot{x} = \mathrm{div}_x T + J^{-1} mg$$

Now, $J^{-1}m = \rho$ (by defn) & $\ddot{x} = \dot{v} = \frac{Dv}{Dt} = v_t + v \cdot \nabla v$

Substn gives the asserted 2nd eqn (cons of momentum).