Mechanics – Lecture 3 (end), 2/13/2019

Last ½-hour of 2/13 lecture started our discussion of 3D (nonlinear) elasticity. These notes go only as far as we did in class (we'll continue this topic on 2/20).

Recommended reading to support my treatment of nonlinear elasticity:

- 1st 20 pages of the book by Marsden + Hughes
- chapter 5 of the book by Howell - Kozyreff - Ockendon

Big picture: we need to address

a) strain (ie description of the stretching or shrinking assoc to a given deformation)

b) stress (ie description of the forces by which one part of a body acts on the part next to it)

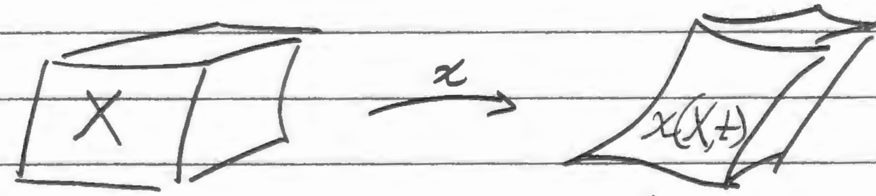
c) balance laws (as usual, assoc to conservation of linear + angular momentum)

d) constitutive laws (relation b'tween stress + strain, which characterizes the material under consideration)

These notes address only (a) [and we'll continue even with (d) next week]

<u>About strain</u> (analysis of deformation; "kinematics"; closely tied to differential geometry)

Basic picture:



$\Omega \subset \mathbb{R}^3$
reference ("Lagrangian") coordinates

deformed ("Eulerian") coordinates

$$X = \{X_\alpha\}_{\alpha=1}^{3}, \quad x = \{x_i\}_{i=1}^{3}$$

conventions on coord labeling help us avoid confusion b'tween Eulerian + Lagrangian vars.

$$F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha} = \text{differential of map } X \to x(X) = \text{"deformation gradient"}$$

Note: The map $X \to x(X)$ should be

injective (no interpenetration of matter) so we expect $\det F > 0$.

<u>Polar decomposition</u>: if $\det F > 0$ then F can be written uniquely as

$$F = RU$$

where R is an orientation-preserving rotation and U is a pos def symmetric matrix. Note that

$$U = (F^T F)^{1/2}$$

Eigenvalues of U are "principal stretches" (eigenvectors are "principal directions").

Nonlinear strain is $U - I$ (or maybe $F^T F - I$, depending from author to author).

<u>Main point</u>: locally, any deformation is like a pos def symm map composed with a rotation.

<u>Question 1</u>: Which matrix-valued fns $F_{i\alpha}(X)$ are deformation gradients?

Answer is standard: in a simply connected domain, $F_{i\alpha} = \partial x_i / \partial X_\alpha$ iff each row is curl-free.

(Note: when studying dislocations in crystals, requirement that $F = \partial x / \partial X$ is relaxed; presence of nonzero curl reflects presence of dislocations in the lattice.)

<u>Question 2</u>: Which matrix-valued functions $g_{\alpha\beta}(X)$ arise as $(F^T F)_{\alpha\beta} = \sum_{i=1}^{3} F_{\alpha i} F_{i\beta}$ for some $F_{i\alpha} = \partial x_i / \partial X_\alpha$?

Answer is more subtle, but still classical: $F^T F$ is a <u>Riemannian metric</u> (namely, the standard metric of $\mathbb{R}^3$, expressed in local coords X rather than Euclidian coords x, since $dx = F\,dX \Rightarrow |dx|^2 = |F\,dX|^2 = \langle F^T F\,dX, dX\rangle$).

<u>In language of differential geometry</u>: a pos def symmetric matrix-valued fn $g_{\alpha\beta}(X)$ can be expressed as $F^T F$ iff it is a "flat metric" (ie the standard metric on $\mathbb{R}^3$ expressed in a weird coordinate system). <u>Basic fact from</u>

differential geometry: $g_{\alpha\beta}$ is flat (locally) iff its "Riemann curvature tensor" vanishes. (This is a nonlinear pde system involving $g_{\alpha\beta}$ and its first + 2nd derivatives.)

Question 3: Which deformations do no stretching or shrinking?

Answer: strain $= 0 \iff F^T F = I$
$\iff F = \partial x / \partial X$ is a rotation-valued function of X.

Assuming a little smoothness, this $\Rightarrow$ $x(X)$ is a rigid motion (This is another classical result from differential geometry).

[We'll continue next week with a couple more questions/answers about strain, then discn of stress, etc]