Mechanics - Lecture 1, 1/30/2019

1st of about 7 lectures on elasticity. The big picture: elasticity is concerned with

stress (forces)

strain (stretching + shrinking; more generally, description of deformation)

constitutive relations (what stress is produced by a given strain - this describes a material's elastic response)

balance laws (conservation of linear momentum + cons. of angular momentum => eqns of ~~elasto~~ elasto-dynamics, or [as a special case] eqns of elastostatic equilibrium)

elastic energy (special form of constit. reln in 3D, but no loss of generality in 1D, leads to links with calculus of varns)

Examples of things we might model this way:

- "elastic strings" - eg rubber bands, or steel cables [if bending resistance is not important]
- "elastic rods" eg a plastic ruler or a flexible cylindrical rod [they resist bending + twisting]
- rubber - nearly incompressible, sustains large deformation w/o fracture, interesting both as 3D solid + also as 2D membrane (eg balloons, or a sheet wrapped around a curved object)

Note: most materials are inelastic at suff'tly large deformation or stress, sometimes importantly so. Some types of inelastic behavior:

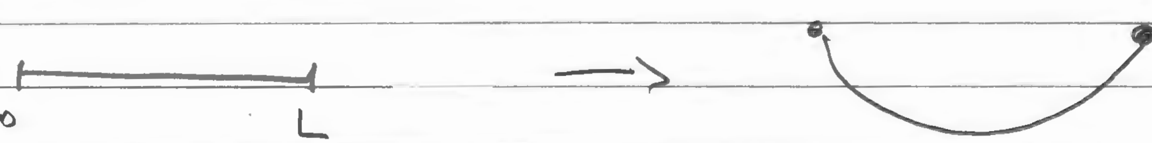
a) plasticity — large stress induces permanent deformation (easy to see in an aluminum paper clip)

b) viscoelasticity - force depends on deformation rate, seen eg in automobile shock absorbers, and in "silly putty" (viscous effects tend to dampen elastic vibrations)

c) fracture (a complicated field, mostly about crack propagation)

Some people would begin with linear elasticity (advantage: linear pde's). Others with fully nonlinear elasticity (advantage: suitable for large deformations). We'll do both, but later.

I prefer to start with elastic strings because the math is easy yet it leads to good intuition, good exercises, and we can already see connections with the Calculus of Variations. Good source: Antman's book chaps 2+3 (of course he does much more).



reference state of string, $0<s<L$ (parametrized by arclength, "unstretched")

deformed state of string at time t $\vec{r}(s,t) \in R^3$

$$\left|\frac{d\vec{r}}{ds}\right| = \nu(s,t) = \text{"stretch ratio" or "strain"}$$

$\nu > 1 \iff$ tension

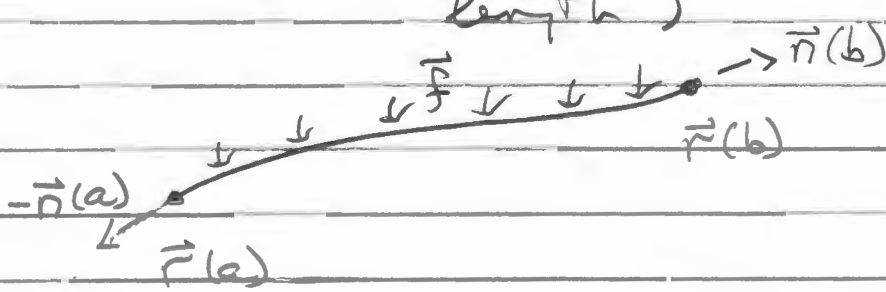
$\nu < 1 \iff$ compression

(Actually you can't really compress a string: since it has no resistance to bending it will buckle instead. Thus any equil one finds with $\nu<1$ should be unstable. You'll find some examples in HW 1.)

<u>Balance of forces</u>: let

$\vec{n}(s_0)$ = force exerted by part of string at $s>s_0$ on part assoc to $s<s_0$

$\int_{s_0}^{s_1} \vec{f}(s)\,ds$ = total "body force" on part of string btwn s_0 & s_1 (thus: $f(s)$ is force per unit <u>reference</u> length)



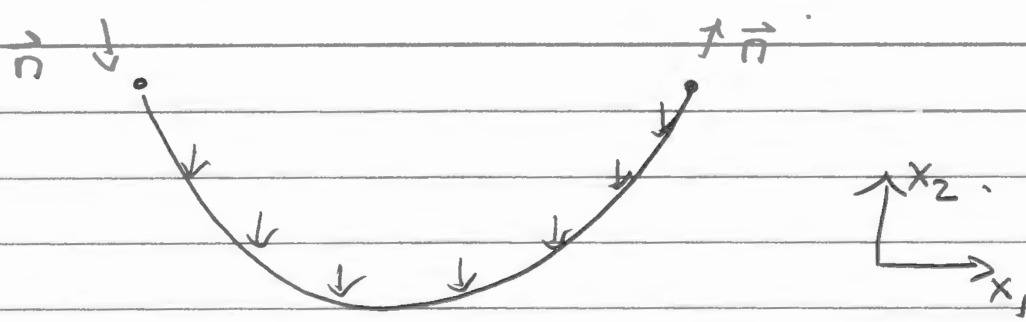
Conservation of momentum ("F = ma") says:

$$\vec{n}(b,t) - \vec{n}(a,t) + \int_a^b \vec{f}(s,t)\,ds = \int_a^b \rho(s)\,\vec{r}_{tt}(s,t)\,ds$$

where $\rho(s)$ = density of string (mass per unit reference length)

Assoc pde is $\quad \vec{n}_s + \vec{f} = \rho\,\vec{r}_{tt}$

[Check correctness of sign: at equil $\vec{r}_{tt} = 0$; for a string hanging under gravity we expect $f_z < 0$ and $\partial_s n_z > 0$ if dirn z is vertical]



Constitutive law: to complete the picture (eg to have a pde that determines the config'n of the string) we need to specify its material response, ie dependence of $\vec{n}$ on $\vec{r}_s$. (The string is "elastic" $\iff$ $\vec{n}$ depends only on $\vec{r}_s$.)
For an elastic string

① $\vec{n}$ is directed tangent to string

② magnitude of $\vec{n}$ depends only on "strain" $|\vec{r}_s| = \nu$ (this is the "principle of frame indifference" in special case of a string)

so

$$\vec{n} = \hat{N}(\nu(s,t)) \frac{\vec{r}_s}{|\vec{r}_s|}$$

(assuming the string is uniform), where $\hat{N}(\cdot)$ is a suitable function from $\mathbb{R}_+$ to $\mathbb{R}$. Expect

$\hat{N}(1) = 0$ (reference string was in its natural, stress-free state)

$\hat{N}(\nu) \to \infty$ as $\nu \to \infty$
$\to -\infty$ as $\nu \to 0$

[Rmk: for a <u>viscoelastic</u> string, the constitutive law would instead be rate-dependent: $\vec{n} = \hat{N}(\nu, \nu_t)\, \vec{r}_s / |\vec{r}_s|$.]

<u>Eqn of equil</u> for string is thus

$$\vec{n}_s + \vec{f} = 0, \quad \vec{n} = \hat{N}(|\vec{r}_s|)\ \vec{r}_s/|\vec{r}_s|$$

(a 2nd order nonlinear ode). It must be supplemented by <u>boundary conditions</u>, which typically specify either $\vec{r}$ or $\vec{n}$ (but not both) at each endpt. (Other bc are possible, eg one end of string could be attached to a surface but slide freely along it.)

When is there an assoc <u>variational principle</u>? In other words, when are equil. states the critical pts of some functional? Var'l prins are important

a) They often exist, + help us analyze problems (and solve them numerically)

b) getting bdry conds right is often easier using the variational principle

c) The shed light on elastic stability (to be explained shortly)

For elastic string it's natural to seek var'l prin involving

$$F[\vec{r}] = \int_0^L W(|\vec{r}_s|)\,ds + \int_0^L V(\vec{r}(s))\,ds.$$

(Why $W(|\vec{r}_s|)$ not $W(\vec{r}_s)$? Because rotating string in space doesn't change its mechanical state; this is the "principle of frame indifference" at the variational level.)

Let's find EL eqn for F, assuming the bc is of "displacement type": $\vec{r}(0) = \vec{a}$, $\vec{r}(L) = \vec{b}$.

Notation: $\delta F = 0$ means $\frac{d}{dt}\Big|_{t=0} F[\vec{r} + t\vec{g}] = 0$

whenever $\vec{g}(0) = 0$ + $\vec{g}(L) = 0$.

Differentiate under integral to get assoc pde (we assume $|\vec{r}_s|$ stays away from 0 + ∞ so this is justified):

$$\frac{d}{dt}\Big|_{t=0} F(\vec{r} + t\vec{g}) = \int_0^L \langle W'(|\vec{r}_s|)\frac{\vec{r}_s}{|\vec{r}_s|}, \vec{g}_s\rangle\,ds$$

$$+ \int_0^L \langle \nabla V(\vec{r}(s)), g\rangle\,ds$$

But 1st term equals

$$-\int_0^L \langle \left(W'(|\vec{r}_s|)\frac{\vec{r}_s}{|\vec{r}_s|}\right)_s, \vec{g}\rangle\,ds$$

(using bc $\vec{g}(0) = \vec{g}(L) = 0$). Now, if a fn $\vec{\varphi}(s)$ satisfies

$$\int_0^L \langle \vec{\varphi}(s), g(s) \rangle = 0 \quad \text{whenever } \vec{g}(0) = \vec{g}(L) = 0$$

Then we must have $\vec{\varphi}(s) \equiv 0$ (Exercise!).
So

$$\delta F = 0 \iff \frac{d}{ds}\left(W'(|\vec{r}_s|) \frac{\vec{r}_s}{|\vec{r}_s|} \right) - \nabla V(\vec{r}(s)) = 0$$

<u>Conclusion</u>: equil eqn of elastic string has a var'l prin iff the force is "conservative", ie

$$\vec{f}(s) = -\nabla V(\vec{r}(s))$$

for some scalar-valued function $V : \mathbb{R}^3 \to \mathbb{R}$

(Note: gravity is conservative!) The assoc elastic energy W satisfies

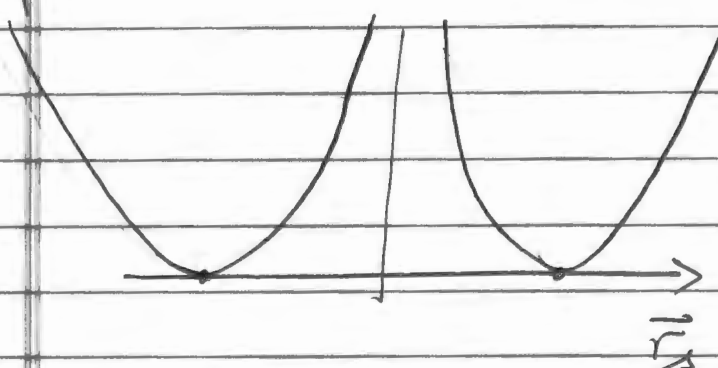
$$W'(\nu) = \hat{N}(\nu), \quad \text{ie } W(\nu) = \int_1^\nu \hat{N}(\tau)\, d\tau$$

<u>A brief digression</u>, looking more closely at this class of var'l problems: The elastic

energy density

$$\vec{r}_s \longrightarrow W(|\vec{r}_s|)$$

is always noncenvex; it is minimized exactly at $|\vec{r}_s| = 1$



graph of W = surface of revolution obtained by rotating figure about vertical axis.

There's a convex problem below this one, agreeing with it when $|\vec{r}_s| \geq 1$, namely

$$\text{"convexification of } W\text{"} = \begin{cases} W(|\vec{r}_s|) & \text{if } |\vec{r}_s| \geq 1 \\ 0 & \text{if } |\vec{r}_s| < 1 \end{cases}$$

Replacing $\int_0^L W(|\vec{r}_s|)\,ds$ by its convexification doesn't change the min value of the functional, but it eliminates lots of critical pts (in some problems), and if resulting solution has $|\vec{r}_s| < 1$ then we have solved a different problem from the original one.

A little more about nonconvexity and why it might matter: consider analogous 1D var'l pbm

$$\int_0^1 (u_x^2-1)^2\,dx \qquad \text{with } u:(0,1)\to\mathbb{R}$$

(which obviously prefers $u_x=\pm 1$). Depending on details, a nonconvex var'l pbm might not achieve its min

$$\text{eg} \quad \min_{\substack{u(0)=0\\ u(1)=0}} \int_0^1 (u_x^2-1)^2 + u^2\,dx$$

has min value 0, but it's not attained.

Returning to our F: if a min-energy state exists then it must not use compressive states (since intro of buckling or "wiggles" would relieve the compression + thereby decrease the energy, while barely changing the term involving V). So: solving pbm with "convexified" W suffices to find min energy states of orig pbm as well (if there are any).

Exercise related to this: show that for an elastic string loaded by gravity, with displacement bdry conds, there *is* a (tensile) min energy state, provided that $W(\nu)$ is convex for $\nu > 1$ (ie $\hat{N}'(\nu) > 0$ for $\nu > 1$, ie more stretching $\Rightarrow$ more stress).

Now let's explain why *local* min of elastic energy represent *stable* equilibria. Assume var'l structure as discussed above, and dead bc (for simplicity). Then eqns of *elastodynamics* imply that

$$\text{kinetic} + \text{elastic} + \text{potential energy} = \text{const}$$

ie

$$\frac{d}{dt}\left\{ \int_0^L W(|\vec{r}_s|) + V(\vec{r}(s))\, ds + \int_0^L \tfrac{1}{2}\rho|\vec{r}_t|^2\, ds \right\}$$

$$= \int_0^L \langle \vec{n}(s,t), \vec{r}_{st}\rangle - \langle \vec{f}, \vec{r}_t\rangle + \rho\langle \vec{r}_{tt}, \vec{r}_t\rangle$$

$$= \int_0^L (-\vec{n}_s - \vec{f} + \rho\vec{r}_{tt}) \cdot \vec{r}_t\, ds$$

$$= 0$$

(Looking ahead: when there's no damping, elastodynamics is *Hamiltonian*, and the evolution preserves the value of the Hamiltonian.)

For a viscoelastic string (with a suitable constitutive law) the analogous calcn would give

$$\frac{d}{dt}\left[\text{kinetic} + \text{elastic} + \text{potential energy}\right] \leq 0$$

with strict ineq unless $\vec{r}_t = 0$.

Conclude: in presence of viscoelastic damping

- a saddle pt of potential + elastic energy cannot be stable
- a local min of potential + elastic energy will be stable, provided that there's an "energy well".

[Digression: The term "local min" here is sloppy. Choice of topology matters, for nonconvex problems! For example, in var'l prbm

$$\Phi(v) = \int_0^1 (v_x^2 - 1)^2 \, dx \qquad \text{where } v:[0,1] \to \mathbb{R}$$

with bc $v(0) = 0$

$v(1) = \alpha$

we can ask: is $v_*(x) = \alpha x$ a local min of Φ?

If α is chosen st $W(\xi) = (\xi^2-1)^2$ has $W''(\alpha) > 0$
then answer is: $v_\gamma(x)$ is a local min in C^1 topology, but it is not a local min in L^∞ topology.

Oh, So, in the bullets on pg 1.13, which topology is appropriate + why? I leave this as food for thought.]

What types of bvp might we want to solve for an elastic string? HW 1 has a few:

① "catenary problem" – a string suspended from its endpts, pulled down by gravity. What curve does it achieve?

② "suspension bridge problem" – like ① but load is now per unit horizontal distance (the roadway)

③ "pressurized ring" – a simple closed curve under uniform pressure should form a circle (if string is uniform)