Mechanics – Lecture 9 – 4/3/2019

Key lessons from last lecture: to go beyond the simplest examples (such as Newton's laws for interacting particles) and to permit considering arbitrary coord systems, the Lagrangian viewpoint

$$\text{eqns of mechanics} \iff \text{EL eqns of} \int_{t_1}^{t_2} L(q,\dot q)\,dt$$

are convenient.

But we'll also see (today) that a different "Hamiltonian viewpt" has it's own advantages; that viewpt says

$$\text{eqns of mechanics} \iff \dot p_i = \frac{-\partial H}{\partial q_i}, \quad \dot q_i = \frac{\partial H}{\partial p_i}$$

where $H(q,p) = \sup_{\xi} \left[\langle p, \xi \rangle - L(q,\xi) \right]$

is the Fenchel transform of $L(q,\dot q)$ wrt $\dot q$; note the consequence

$$H(q,p) = \langle p, \dot q \rangle - L(q,\dot q)$$

where $\dot q$ is the (unique) soln of $\frac{\partial L}{\partial \dot q}(q,\dot q) = p$.

A basic example to keep in mind is

$$(*) \quad L = \frac{1}{2} \sum{}' a_{ij}(q)\, \dot{q}_i \dot{q}_j - U(q)$$

where $a_{ij}(q)$ is pos definite + symmetric (this arises eg for interacting particles in a non-Euclidean coord system). Then

$$(**) \quad H = \frac{1}{2} \sum{}' [a^{-1}(q)]_{ij}\, p_i p_j + U(q)$$

where $a^{-1}(q)$ is the matrix inverse to $a(q)$. [Exercise: show, by direct calculation, that when L has the form $(*)$ and $q(t)$ solves EL eqn of $\int L(q,\dot{q})\,dt$ then $q(t)$ and $p_i(t) = \sum{}' a_{ij}(q)\, \dot{q}_j(t)$ solve

$$\dot{p}_i = \frac{-\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad .]$$

Which $L(q,\dot{q})$ are allowed? We need to assume that L is strictly convex in $\dot{q}$, with faster-than linear growth at ∞. (We'll return to discn of this, and discn of the Fenchel transform, a little later.)

Before delving into relationships btwn Lagrangian + Hamiltonian viewpts, let's see some of the advantages of each (which are interesting even for interacting particle in Euclidean coordinates).

<u>A key advantage of Lagrangian viewpt is to connect invariances of L with conservation laws.</u>

General version of this is "Noether's thm." To keep things concrete, I'll just do some simple cases.

<u>Example 1</u>: Reduction of a 2D particle in a central force field to an ODE for $r(t)$ only: Suppose Newtonian particle in $\mathbb{R}^2$ experiences force assoc potential $U(r)$ with $r = |x|$. Using polar coords, and remembering that $\dot{x} = \dot{r} e_r + \dot{\theta} r e_\theta$,

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

and EL eqn for $\int L(r, \theta, \dot{r}, \dot{\theta})\, dt$ becomes

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right), \qquad \underbrace{\cancel{\frac{\partial L}{\partial \theta}}}_{0} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}$$

$$\Longrightarrow \begin{cases} -U'(r) + m r \dot{\theta}^2 = \frac{d}{dt}(m \dot{r}) \\ 0 = \frac{d}{dt}(m r^2 \dot{\theta}) \end{cases}$$

Assuming $m = $ const, this gives

$$m r^2 \dot{\theta} = c_0 \quad \Rightarrow \quad \dot{\theta} = \frac{c_0}{m r^2}$$

$$\Rightarrow \quad m \ddot{r} = -U'(r) + \frac{c_1}{r^3}$$

where $c_1 = c_0^2/m$. In particular, c_1 is constant (determined by m and the initial values of r and $\dot{\theta}$).

Essential mechanism of this calculation: L was indep of θ, leading to "conservation law" $r^2 \dot{\theta} = \text{const}$.

Example 2: If $L(q, \dot{q})$ has no explicit dependence on time, trajectories "conserve energy" in the sense that

$$E = \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$$

is constant in time. (Note that if $L = T - U$ with $T = \frac{1}{2} \sum a_{ij}(q) \dot{q}_i \dot{q}_j$ then

$$\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T$$

so $E = 2T - (T - U) = T + U = H$, as expected.)

Pf: EL eqn for $\int L(q, \dot{q})\, dt$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

so (since L doesn't depend on t)

$$\frac{d}{dt} L = \sum' \frac{\partial L}{\partial q_i}\dot{q}_i + \sum' \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= \sum' \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum' \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= \frac{d}{dt}\left(\sum' \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

<u>Example 3</u>: Cons of linear momentum comes similarly from transl invariance, ie hypoth that for $\vec{a} \in \mathbb{R}^3$, $L(q_1+\vec{a}, q_2+\vec{a}, \cdots, q_N+\vec{a}, \dot{q}_1, \cdots, \dot{q}_N) = L(q_1, \cdots, q_N; \dot{q}_1, \cdots \dot{q}_N)$. In fact if

$$\frac{d}{dt} L(q_1+t\vec{a}, \cdots, q_N+t\vec{a}; \dot{q}_1, \cdots, \dot{q}_N) = 0$$

for any $\vec{a} \in \mathbb{R}^3$ then

$$\sum_{i=1}^{N}{}' \frac{\partial L}{\partial q_i} = 0 \quad \text{as a vector in } \mathbb{R}^3.$$

Combine with EL for $\int L(q, \dot{q})\,dt$ to get

$$\sum_{i=1}^{N}{}' \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

ie $\sum_{i=1}^{N}{}' \partial L/\partial \dot{q}_i$ is constant along trajectories.

<u>Example 4</u>: Cons of angular momentum follows similarly from rotational invariance, ie from

$$L(q_1, \dots q_N; \dot{q}_1, \dots, \dot{q}_N) = L(Rq_1, \dots, Rq_N; R\dot{q}_1 \dots R\dot{q}_N)$$

for any rotation R. (For a proof see eg §9 of Landau + Lifschitz.)

<u>Example 5</u> (really a special case of the 4^{th}): for particles in $\mathbb{R}^3$ with forces $f_i = -\partial U/\partial x_i$, if U is invariant under rotn abt $\vec{e}$ axis then $\vec{e}$ component of ang. momentum is conserved.

<u>Pf</u>: $\frac{d}{dt} \sum_i' [x_i \times (m_i \dot{x}_i)] \cdot \vec{e}$

$$= \frac{d}{dt} \sum_i \left(x_i \times \frac{\partial L}{\partial \dot{x}_i}\right) \cdot \vec{e}$$

$$= \sum_i' \Big(\underbrace{\dot{x}_i \times \cancel{\frac{\partial L}{\partial \dot{x}_i}}}_{0}\Big) \cdot \vec{e} + \underbrace{\sum_i' \left(x_i \times \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}\right) \cdot \vec{e}}_{\sum_i' \left(x_i \times \frac{\partial L}{\partial x_i}\right) \cdot \vec{e}}$$

since $\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i$

Now use fact: if $\varphi_t(x)$ = rotn of x about $\vec{e}$ axis, $\frac{d}{dt} \varphi_t(x) = \vec{e} \times \vec{x}$. So hypothesis on L says

$$\sum' \frac{\partial L}{\partial x_i} \cdot (\vec{e} \times x_i) = 0$$

Now finally use vector identity $a\cdot(b\times c) = -b\cdot(a\times c)$ [they're both $\det(a,b,c)$] to conclude that

$$\sum_i{}' \left(x_i \times \frac{\partial L}{\partial x_i}\right)\cdot\vec{e} = 0 .$$

So $\sum_i{}' [x_i \times (m_i \dot{x}_i)]\cdot\vec{e}$ = const along trajectories.

The Hamiltonian viewpoint also has some key advantages. They apply to any Hamiltonian system

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

(and they're nontrivial even for Euclidean particles interacting by potential $U(q_1, \dots, q_N)$ for which $p_i = m_i \dot{q}_i$ and $H = \frac{1}{2}\sum{}' \frac{1}{m_i}|p_i|^2 + U$)

1st consequence: H is constant along trajectories

$$\frac{d}{dt}H = \sum \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i = 0$$

(assuming H is a fn of q and p only, with no additional dependence on t). This is of course the same conservation-of-energy law we got before from the Lagrangian viewpt.

<u>2nd consequence:</u> Liouville's Theorem: in the (q, p) coordinates, the flow assoc to our evolution is <u>volume-preserving</u>

<u>Pf:</u> For any flow we can consider its "infinitesimal generator"

$$\text{image of } \vec{x} \text{ after time } t = \vec{x} + \vec{f}(x)\,t + O(t^2).$$

and the flow is vol-preserving iff $\operatorname{div} \vec{f} = 0$, since

$$\frac{d}{dt}\Big|_{t=0} (\text{vol of image of } D) = \int_D \frac{d}{dt}\Big|_{t=0} \det(I + t\nabla f)\,dx$$

$$= \int_D \operatorname{div} \vec{f}\,dx ;$$

Apply this to Hamiltonian flow using $\vec{x} = (\vec{q}, \vec{p})$ and $\vec{f} = \left(\frac{\partial H}{\partial p}, \frac{-\partial H}{\partial q}\right)$:

$$\operatorname{div} f = \sum \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i}\right) = 0.$$

Liouville's thm has some surprising consequences, eg

<u>Poincaré's recurrence thm</u>: if g is a vol preserving map (eg the time-one map of a

Hamiltonian flow) and $g(D) = D$ for some set D of finite volume, then it is "recurrent" in sense that

> for any set B of pos measure [e.g. a tiny ball], $\exists\, x_0 \in B$ s.t. $g^n(x_0)$ is again in B for some $n < \infty$.

[Instructive examples: rots of S^1 by a rational or irrational angle.]

Pf of recurrence: clearly $B, g(B), g^2(B)$, etc cannot all be disj't, so $\exists\, x_1 \in g^k(B) \cap g^l(B)$ for some $l < k$. Then $x_0 = g^{-k}(x_1)$ satisfies $x_0 \in B \cap g^{l-k}(B)$. So

$$x_0 \in B \text{ and } g^{k-l}(x_0) \in B,$$

as claimed.

Typical mechanical consequence: consider motion of a ball in an asymmetric bowl, under action of gravity

Region of phase space st $T+U \le$ constant is invariant and has finite volume. So ball returns to almost its initial position and velocity.

One more consequence: "dimension reduction" - if the Lagrangian is indep of q_1 then so is the Hamiltonian. As a result p_1 is constant & problem reduces to Hamilton's eqns in $(q_2, \dots, q_N; p_2, \dots, p_N)$.

In fact, $\frac{dp_1}{dt} = \frac{-\partial H}{\partial q_1} = 0$ by hypoth, so $p_1 = $ const.

So we can solve $\dot{p}_j = \frac{-\partial H}{\partial q_j}, \dot{q}_j = \frac{-\partial H}{\partial p_j}$ $(j \ge 2)$

by substituting the (constant) value of p_1 into H. Finally, get $q_1(t)$ at the end by integrating

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}$$

along the resulting path.

This argument can be repeated. So: if L and H depend on just one "spatial" variable then the evolution can be reduced to phase plane analysis.

(Note: our 1st Lagrangian example — a planar particle in a central force field — can also be viewed as an example of this principle.)

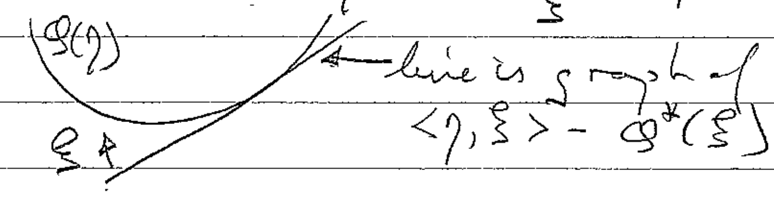
We haven't yet discussed *why* the Lagrangian + Hamiltonian viewpts are equivalent (except for interacting particles, where it's an elementary calculation).

Warming up for that discussion, we need to discuss the Fenchel transform (also called Legendre transform) of a convex fn φ.

- A convex fn is a sup of linear fns. Thus, for each slope we can adjust the intercept so assoc plane touches the graph. Envelope of these planes must be the given convex fn.

- Eqn-based version of previous bullet:

$$\varphi(\eta) \text{ convex} \Rightarrow \varphi(\eta) = \max_{\xi} \langle \eta, \xi \rangle - \varphi^*(\xi)$$



What is $\varphi^*(\xi)$? Well, by convexity

$$\varphi(\eta) \geq \langle \eta, \xi \rangle - \varphi^*(\xi) \text{ with equality for each } \xi \text{ at some } \eta.$$

so

$$\varphi^*(\xi) \geq \langle \eta, \xi \rangle - \varphi(\eta) \text{ with equality for each } \xi \text{ at some } \eta.$$

so

$$\varphi^*(\xi) = \max_{\eta} \langle \eta, \xi \rangle - \varphi(\eta)$$

To be sure all slopes occur, one needs $\varphi(\eta)/|\eta| \to \infty$ as $|\eta| \to \infty$ ("superlinear growth").

Relevance to Lagr vs Hamilt mechanics:

$$H(q,p) = \max_{\dot q} \langle \dot q, p \rangle - L(q, \dot q).$$

= Fenchel transform of L wrt $\dot q$ (holding q fixed).

We want

① $p_i = \frac{\partial L}{\partial \dot q_i}$ determines a well-defined change of coords $(q, \dot q) \to (q, p)$

② we can recover the Lagrangian from

the Hamiltonian by

$$L(q,\dot{q}) = \max_p \langle \dot{q}, p\rangle - H(q,p)$$

③ we can get $\dot{q}$ as fn of p and q by

$$\dot{q} = \frac{\partial H}{\partial p_i}$$

Returning to a general convex fn $\varphi(\eta)$ with

$$\frac{\varphi(\eta)}{|\eta|} \to \infty \quad \text{as } |\xi| \to \infty$$

(note: the growth cond^n assures that $\varphi^*(\xi) = \max_\eta \langle \xi, \eta\rangle - \varphi(\eta)$ is finite for any ξ)

properties ①–③ follow from the following two facts

(A) φ^* is convex, and $\frac{\varphi^*(\xi)}{|\xi|} \to \infty$ as $|\xi| \to \infty$

(B) $\varphi^{**} = \varphi$.

<u>Proof of (A)</u>: φ^* = max of linear fns, so it's certainly convex. Taking $\eta = \lambda \xi/|\xi|$ as a test choice gives

$$\varphi_*(\xi) \geq \lambda|\xi| - \varphi(\lambda \xi/|\xi|)$$

$$\Rightarrow \quad \frac{\varphi_*(\xi)}{|\xi|} \geq \lambda - \underbrace{\frac{\min \text{ of } \varphi \text{ on } B_\lambda}{|\xi|}}_{\to 0 \text{ as } |\xi| \to \infty}$$

So

$$\liminf_{|\xi|\to\infty} \frac{\varphi_*(\xi)}{|\xi|} \geq \lambda \quad \text{for any } \lambda \in \mathbb{R}$$

<u>Proof of (B)</u> Clearly $\varphi^*(\xi) + \varphi(\eta) \geq \langle \xi, \eta \rangle$ for all ξ, η. So

$$\varphi(\eta) \geq \langle \xi, \eta \rangle - \varphi^*(\xi)$$

whence

$$\varphi \geq \varphi^{**}$$

For the reverse, observe that

$$\varphi^*(\xi) = \langle \xi, \eta \rangle - \varphi(\eta) \quad \text{when } \nabla_\eta \varphi = \xi.$$

So $\qquad \varphi(\eta) = \langle \xi, \eta \rangle - \varphi^*(\xi) \quad \text{when } \nabla_\eta \varphi = \xi.$

Thus $\qquad \varphi^{**}(\eta) = \max_\xi \langle \xi, \eta \rangle - \varphi^*(\xi)$

$$\geq \langle \nabla_\eta \varphi, \eta \rangle - \varphi^*(\nabla_\eta \varphi) = \varphi(\eta)$$

So $\varphi^{**} = \varphi$, as claimed.

One more note: hypothesis that $L(q,\dot q)$ is strictly convex in $\dot q$ is natural to make $\int L(q,\dot q)\,dt$ well-behaved as a calculus of var'ns problem. In particular

- if convexity fails then $\min \int_{t_1}^{t_2} L(q,\dot q)\,dt$ might not be achieved, since oscillatory fns of time might be preferred instead; an elementary example is

$$\int_0^1 u^2 + (\dot u^2-1)^2\,dt$$

for which the min value is 0, but it isn't achieved

- strict convexity permits proving higher reg'y of solns of the EL eqns; formally, if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = \frac{\partial L}{\partial q_i}$$

then

$$\sum_j \frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\,\ddot q_j = \frac{\partial L}{\partial q_i} - \sum \frac{\partial^2 L}{\partial \dot q_i \partial q_l}\,\dot q_l$$

If $\frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}$ is invertible then we can

differentiate this eqn wrt t to (formally) determine higher derivatives of $q(t)$.
This calculation can be justified, leading to conclusion that soln is C^∞ in t.