Mechanics - Lecture 8 - 3/27/2019

New topic today: 1st of 3 lectures on "classical mechanics," emphasizing links to pde and the calculus of variations.

Major goals include:

- some basic examples - to gain intuition + to see scope of theory + to see links with physics and pde.
- The Lagrangian + Hamiltonian approaches (apparently very different, but equivalent!). Their relative advantages.
- key properties of Hamiltonian systems, eg conservation laws + Liouville's Thm.
- link to pde + optimal control via Hamilton-Jacobi eqns and Pontryagin's max principle.

My syllabus has lots of books. We won't follow any of them linearly, but 1st chapters of Landau/Lifshitz, Arnold, and José/Saletan have lots more examples than we'll have time for in class.

The most basic examples involve motion of interacting particles in $\mathbb{R}^D$ (eg $\mathbb{R}^3$):

$$(*) \quad m_i \ddot{x}_i = f_i$$

m_i = mass of i^{th} particle
$\ddot{x}_i$ = accel of i^{th} particle
f_i = force on i^{th} particle

(here $x_i(t) \in \mathbb{R}^D$). We'll usually focus on conservative forces

$$(**) \quad f_i = -\frac{\partial U}{\partial x_i}$$

where $U = U(x_1, \dots, x_N)$ is fn of positions of the particles

Key feature of Newton's eqns $(*)$ with conservative forces $(**)$:

$$H = \underbrace{\tfrac{1}{2} \sum m_i |\dot{x}_i|^2}_{\text{kinetic energy}} + \underbrace{U}_{\text{potential energy}}$$

is conserved, since

$$\frac{dH}{dt} = \sum m_i \langle \dot{x}_i, \ddot{x}_i \rangle + \sum \langle \frac{\partial U}{\partial x_i}, \dot{x}_i \rangle$$

$$= 0 \quad \text{when } (*) \text{ and } (**) \text{ hold.}$$

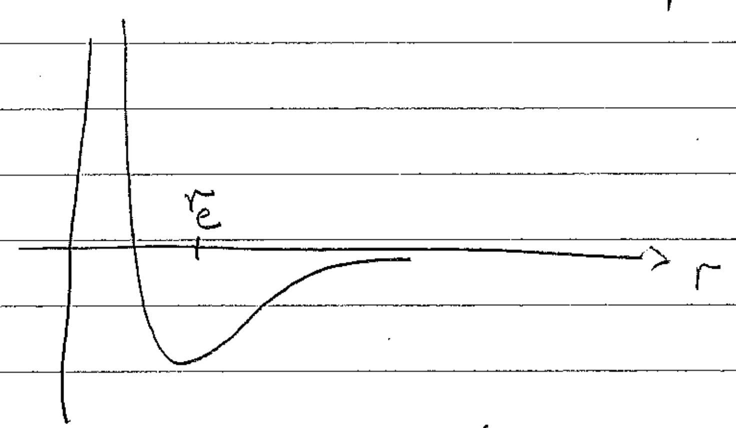
Example 1: many particles interacting by pairwise attraction / repulsion

$$m \ddot{x}_i = - \sum_{j \neq i}' \frac{\partial}{\partial x_i} V(|x_i - x_j|) .$$

Very widely used (eg for modeling fluids): The Lennard-Jones potential

$$V(r) = c\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

where σ, c are constants. Graph looks like



where r_e solves $(\sigma/r_e)^6 = 1/2$. Particles repel when $|x_i - x_j| < r_e$ + attract when $|x_i - x_j| > r_e$, but attraction is negligible when $|x_i - x_j|/r_e \to \infty$ while repulsion is strong as $|x_i - x_j|/r_e \to 0$.

Example 2: nodal values of wave eqn, discretized in space (but not time):

$$\ddot{W}_i = \frac{W_{i+1} + W_{i-1} - 2W_i}{(\Delta X)^2}$$

If wave eqn has $w(x,t)$ defined for $0 < x < 1$ with Dir bc $w = 0$ at $x = 0, 1$ (for example) then discrete version uses $W_1, \dots, W_{N-1}$ with $\Delta x = \frac{1}{N}$ ($W_i(t)$ are scalar-valued), and the laws for $\ddot{W}_1$ and $\ddot{W}_{N-1}$ must take into acct that $W_0 = W_N = 0$.
Here the potential is

$$U = \frac{1}{2} \sum' \left(\frac{W_i - W_{i-1}}{\Delta x} \right)^2$$

Example 2a: nonlinear wave eqns produce similar examples, eg

$$\ddot{W}_i = \frac{W_{i+1} + W_{i-1} - 2W_i}{(\Delta X)^2} + W_i^p$$

is assoc to potential

$$U = \frac{1}{2} \sum' \left(\frac{W_i - W_{i-1}}{\Delta x} \right)^2 - \frac{1}{p+1} \sum' W_i^{p+1}$$

Cont's analogue is of course $W_{tt} = \Delta W + W^p$
which is formally Hamiltonian
with potential $\int \frac{1}{2} |\nabla w|^2 - \frac{1}{p+1} w^{p+1} \, dx$

Example 3: a system of masses and springs produces a Hamiltonian system a lot like that of example 1:

$$m_i \ddot{x}_i = - \sum_{\substack{j \text{ attached} \\ \text{to } i \text{ by spring}}} \frac{\partial}{\partial x_i} V_{ij}(|x_i - x_j|)$$

masses of nodes can be different

each spring can have its own force law

Here $x_i \in \mathbb{R}^3$ (for masses + springs in $\mathbb{R}^3$).

Using a lattice of springs + $\Delta x \to 0$ we can get (nonlinear) elasticity as a limit, much as we got the wave eqn in Example 2a.

Explicitly solvable examples are rare, but worth considering since they provide valuable intuition. Some classes:

a) Linear eqns $\ddot{x} = -Ax$ where $A \geq 0$ is symmetric matrix. Our example 2 was like this (but examples 2a and 3 are not).
Simplest case: in 1D, $x(t) \in \mathbb{R}$ solves $m\ddot{x} = -\alpha x \iff x(t) = C \sin(\omega t + \varphi)$ with $\omega^2 = \alpha/m$

b) a particle in a central force field in $\mathbb{R}^n$ is exactly solvable:

$$m\ddot{x} = -\nabla\varphi(|x|) \quad , \quad x(t) \in \mathbb{R}^N$$

can be reduced to an ODE of form

$$m\ddot{r} = -\partial\psi/\partial r$$

for suitable choice of $\psi(r)$. (This reduction is nontrivial, since $x/|x|$ is not indep of t).

c) any system with just one dof can be understood by phase plane analysis; The "planar pendulum" $\ddot{\theta} = -\sin\theta$ is a classic example

d) The 2-body problem in $\mathbb{R}^3$

$$m_1\ddot{x}_1 = -\frac{\partial U}{\partial x_1} \quad , \quad m_2\ddot{x}_2 = -\frac{\partial U}{\partial x_2}$$

where $U = U(|x_1 - x_2|)$ can reduced to a system with 1 dof (hence fully understood by phase plane analysis). When $U(r) = -k/r$ (gravitational force law) this leads to Kepler's Laws.

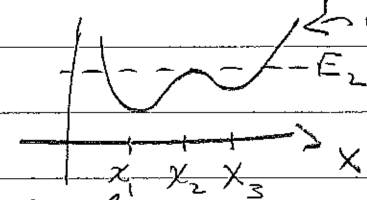
I leave you to read about such examples (Arnold, Landau/Lifshitz, & José/Saletan all have such stuff in 1st chapter or 2).

Let me dwell only on (c): the use of phase plane analysis. If $x(t) \in \mathbb{R}$ and $\ddot{x} = -U'(x)$ then eqn determines a flow in 2D "phase plane" $(x, \dot{x})$. But

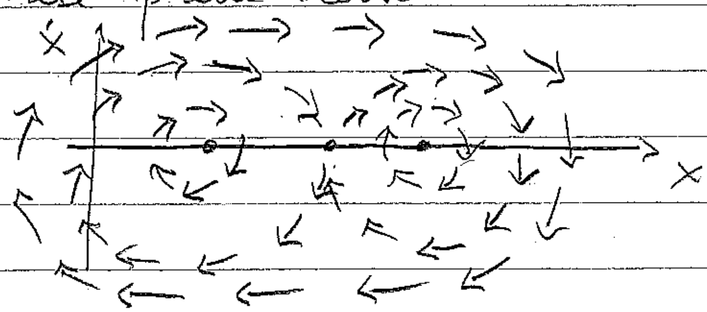
$$H = \frac{1}{2}\dot{x}^2 + U(x) \quad \text{is constant,}$$

so flow evolves along a 1D curve in phase space. This makes visualization easy (though exact formulas may not be available). For example: suppose

$$\ddot{x} = -U'(x) \quad \text{where}$$



Phase plane picture is filled out by trajectories along $\frac{1}{2}\dot{x}^2 + U(x) = E$. When U is as shown, $U'(x) = 0$ has 3 roots; E_2 is the value of U at x_2. Phase plane looks like



x_1 + x_3 (local min of U) are stable;
x_2 (local max of U) is unstable

For $E > E_2$, trajectory surrounds all 3 critical pts.

An interacting particle system is "closed" if the only forces present are those assoc to pairwise interactions (via pairwise potentials $U = \sum' U_{ij}(|x_i - x_j|)$). In such a system

$$m_i \ddot{x}_i = \sum_{j \neq i}' F_{ij} \quad \text{where } F_{ij} \parallel x_i - x_j \text{ and } F_{ji} = -F_{ij}$$

Key property of such systems: <u>linear momentum</u> and <u>angular momentum</u> are conserved.

Here, by defn, <u>linear momentum</u> $= \sum_{i=1}^{N}{}' m_i \dot{x}_i$

and conservation is elementary:

$$\frac{d}{dt} \sum' m_i \dot{x}_i = \sum' m_i \ddot{x}_i = \sum_i{}' \sum_{j \neq i}{}' F_{ij} = 0$$

since $F_{ij} = -F_{ji}$.

Also, by defn, angular momentum $= \sum' x_i \wedge m_i \dot{x}_i$ (motion can be in any $\mathbb{R}^7$ for defn of linear momentum, but must be in $\mathbb{R}^3$ for defn of angular momentum, since we need the vector crossproduct $a \wedge b$). Conservation is again elementary:

$$\frac{d}{dt}\sum_i{}' x_i \wedge m_i \dot{x}_i = \sum_i{}' x_i \wedge m_i \ddot{x}_i$$
$$= \sum_i{}' \sum_{j\neq i}{}' x_i \wedge F_{ij}$$

For any pair $i\neq j$, $F_{ij} = \lambda(x_j - x_i) + F_{ji} = -\lambda(x_j - x_i)$
so
$$x_i \wedge F_{ij} + x_j \wedge F_{ji} = \lambda x_i \wedge (x_j - x_i) - \lambda x_j \wedge (x_j - x_i)$$
$$= \lambda(x_i - x_j)\wedge(x_j - x_i) = 0$$

Thus, entire sum $= 0$.

These laws look like accidents now; we'll get a more fundamental view soon, via the Lagrangian viewpt.

<u>Lagrangian viewpt</u>: Newton's eqns

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}$$ where each $x_i \in \mathbb{R}^3$ and $U = U(x_1, \dots, x_N)$

are identical to EL eqn for the <u>action</u>

$$\int_{t_0}^{t_1} L(q,\dot{q})\,dt$$ where $q = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$
$\dot{q} = (\dot{x}_1, \dots, \dot{x}_N) \in \mathbb{R}^{3N}$

$$L(q,\dot{q}) = \underbrace{\frac{1}{2}\sum{}' m_i |\dot{x}_i|^2}_{T = \text{kinetic energy}} - U(x_1, \dots, x_N).$$

note the minus

Proof is elementary (of course): EL eqn is

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}, \text{ which reduces to}$$

$$-\frac{\partial U}{\partial x_i} = \frac{d}{dt}(m_i \dot{x}_i) = m_i \ddot{x}_i$$

(I have assumed m_i is constant; otherwise correct version of Newton's law would have been $-\frac{\partial U}{\partial x_i} = \frac{d}{dt}(m_i \dot{x}_i)$, not $m_i \ddot{x}_i$.)

<u>Hamiltonian viewpt</u>: in well-chosen local coord system (p_i, q_i) on phase space, the paths we're interested in satisfy the ODE's

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

for some $H = H(q_1, \dots, q_N, p_1, \dots, p_N)$. For our elementary example

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}$$

The choice of coords + H are easy: $p_i = m_i \dot{x}_i$, $q_i = x_i$,

$$H = \underbrace{\frac{1}{2}\sum \frac{1}{m_i}|p_i|^2}_{\text{kinetic energy again}} + U(q_1, \dots, q_N)$$

note the plus sign this time

(Pf is elementary.)

Advantages of Lagrangian viewpt:

A) easy to consider constrained problems (just impose a constraint in the var'l problem; the EL eqn gets a Lagr mult term of course)

B) easy to change coords

C) Lagr viewpt turns mechanics into a calculus of varns problem (but: trajectories may not minimize $\int L(q,\dot{q})$, they are just crit pts; also, note that I haven't said much abt bc).

D) Lagr viewpt lets us see that conservation laws come from symmetries (this was 1st understood systematically by E Noether)

Advantages of Hamiltonian viewpt

1) Immediate consequence is that H = constant along trajectories (it's crucial here that H = fn of q and p only, with no other time dependence)

2) Liouville's thm: in the (q,p) coords,

The evolution is a <u>volume-preserving</u> flow on phase space

3) Poincaré's recurrence theorem (really a result on vol-preserving flows): if g is a vol-pres map (eg time-one map of a Hamiltonian flow) and $g(D)=D$ for some set D of finite volume, then it is "recurrent" in sense that

for any set B of pos measure (eg a tiny ball) $\exists\, x_0 \in B$ st $g^n(x_0)$ is again in B for some $n < \infty$.

Typical mechanical consequence: consider motion of a ball in an asymmetric bowl.

Region of phase space st $T+U \leq$ const is invariant + has finite volume. So ball returns to <u>almost</u> its initial position + velocity.

Explanation of these "advantages" will take a while. Also, we need to discuss: for a general

Lagrangian $L(q,\dot q)$, what is the assoc $H(q,p)$, and what is the relationship between $(q,\dot q)$ and (q,p)?

But first, let's try to capture why the Lagrangian viewpt makes it easy to consider constrained problems and to change coordinates (points A + B on pg 8.11).

<u>Warmup</u>: let's see analogous assertions in a more familiar setting, namely <u>geodesics</u> (which solve the EL eqn for <u>arclength</u>)

- in $\mathbb{R}^n$, arclength $= \int_{t_1}^{t_2} |\dot y(t)|\,dt$. Euler-Lagr eqn (if $|\dot y| \neq 0$) is

$$\frac{d}{dt}\left(\frac{\dot y}{|\dot y|}\right) = 0.$$

Critical pts are thus straight segments. (Arclength is parametrization-independent, so we can choose a constant-speed parametrization $|\dot y| = \text{const}$; then EL becomes $\ddot y = 0$.)

- Now what about geodesics on the unit sphere $|y| = 1$? Introducing Lagr mult $g(t)$ for the ptwise constraint $|y|^2 = 1$, EL eqn is that

$$\int_{t_1}^{t_2} |\dot{y}| + g(t)\,(|y|^2 - 1)\ dt$$

be stationary, ie that

$$\frac{d}{dt}\left(\frac{\dot{y}}{|\dot{y}|}\right) = 2g(t)\,y(t)$$

ie that $\frac{d}{dt}\left(\frac{\dot{y}}{|\dot{y}|}\right)$ is normal to the sphere

[Again, we can choose $|\dot{y}| = \text{const}$ and then the eqn becomes: $\ddot{y}$ is normal to sphere.]

- But maybe we would prefer to use local coordinates on the sphere? If local coords are $(\xi_1, \dots, \xi_{n-1})$ then curve $y(t)$ is represented in local coords by some $\xi(t)$, and

$$|\dot{y}|^2 = \sum' g_{ij}(\xi)\,\dot{\xi}_i\,\dot{\xi}_j$$

where $g_{ij}(\xi)$ [the Riemannian metric] is obtained from the change of variables by chain rule. So in local coords we are considering crit pts of

$$\int_{t_1}^{t_2} \left(\sum' g_{ij}(\xi)\,\dot{\xi}_i\,\dot{\xi}_j\right)^{1/2}$$

Its critical pts are the <u>same</u> as the ones we found before using Lagrange multipliers.

Mechanical analogue is almost the same:

- A free particle in space (no forces) is modelled using $L(\underset{\sim}{q}, \dot{\underset{\sim}{q}}) = \frac{1}{2}|\dot{\underset{\sim}{q}}|^2$

$$\text{crit pt of } \int_{t_1}^{t_2} \tfrac{1}{2}|\dot{\underset{\sim}{q}}|^2 \, dt \iff \ddot{\underset{\sim}{q}} = 0$$

  Of course the particle travels in a straight line; also, $\frac{d}{dt}|\dot{\underset{\sim}{q}}|^2 = \langle \dot{\underset{\sim}{q}}, \ddot{\underset{\sim}{q}} \rangle = 0$ $\Rightarrow$ path is traversed at constant speed. Same soln we got for arclength problem, except that only constant-speed curves arise.

- A marble rolling in a spherical bowl follows a path that's a crit pt of

$$\int_{t_1}^{t_2} \tfrac{1}{2}|\dot{\underset{\sim}{q}}|^2 \qquad \text{constrained by } |\underset{\sim}{q}(t)|^2 \equiv 1$$

(I assume the sphere has radius 1). By method of Lagr multipliers, condition is

$$\ddot{\underset{\sim}{q}} \text{ is a multiple of } q(t)$$
(ie is normal to the sphere).

Note that $\frac{d}{dt}|\dot{q}|^2 = \langle \dot{q}, \ddot{q} \rangle = 0$, so speed is still constant; we again pick up same solution as for the geometric problem (a geodesic) but parametrized with constant speed.

- In local coords, the same problem leads instead to the EL eqn for

$$\frac{1}{2}\int_{t_1}^{t_2} \sum g_{ij}(\xi)\, \dot{\xi}_i \dot{\xi}_j \, dt$$

Note: my choice to work with a sphere was just to keep the geometry transparent and familiar. The same calculation actually works for any codimension-one hypersurface.

[You might well ask: if the local-coordinate Lagrangian is $L(\xi, \dot{\xi}) = \frac{1}{2}\sum g_{ij}(\xi)\dot{\xi}_i \dot{\xi}_j$, what is the associated H? The answer is:

$$H(\xi, p) = \frac{1}{2}\langle g^{-1}(\xi)p, p\rangle$$

$$p_i = \sum g_{ij}(\xi)\dot{\xi}_j$$

where g^{-1} is the matrix inverse of g. This is a special case of the corresp between L + H, to be discussed in the next lecture.]