Derivative Securities – Fall 2012 – Section 9
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This section discusses the pricing of options on interest-based instruments – i.e. pricing of bond options, caps, floors, and swaptions. We focus on two approaches: (i) Black's model, and (ii) trees. Briefly: we're taking the same methods developed earlier this semester for options on a stock or forward price, and applying them to interest-based instruments.

The material discussed here can be found in chapters 27, 28, and 30 of of Hull (8th edn). I'll also take some examples from the book *Implementing Derivatives Models* by Clewlow and Strickland, Wiley, 1998 (which is on reserve in the Courant library, though it was not listed on my syllabus).

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Black's model. Recall the formulas derived in Section 5 for the value of a put or call on a forward price:

$$\begin{aligned} c[\mathcal{F}_0, T; K] &= e^{-rT}\left[\mathcal{F}_0 N(d_1) - KN(d_2)\right] \\ p[\mathcal{F}_0, T; K] &= e^{-rT}\left[KN(-d_2) - \mathcal{F}_0 N(-d_1)\right] \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\log(\mathcal{F}_0/K) + \tfrac{1}{2}\sigma^2 T\right] \\ d_2 &= \frac{1}{\sigma\sqrt{T}}\left[\log(\mathcal{F}_0/K) - \tfrac{1}{2}\sigma^2 T\right] = d_1 - \sigma\sqrt{T}. \end{aligned}$$

Black's model values interest-based instruments using almost the same formulas, suitably interpreted. One important difference: since the interest rate is no longer constant, we replace the discount factor e^{-rT} by $B(0,T)$.

The essence of Black's model is this: consider an option with maturity T, whose payoff $\phi(V_T)$ is determined by the value V_T of some interest-related instrument (a discount rate, a term rate, etc). For example, in the case of a call $\phi(V_T) = (V_T - K)_+$. Black's model stipulates that

(a) the value of the option today is its discounted expected payoff.

No surprise there – it's the same principle we've been using all this time for valuing options on stocks. If the payoff occurs at time T then the discount factor is $B(0,T)$ so statement (a) means

$$\text{option value} = B(0,T)E_*[\phi(V_T)].$$

We write E_* rather than E_{RN} because in the stochastic interest rate setting this is *not* the risk-neutral expectation; we'll explain why E_* is different from the risk-neutral expectation later on. For the moment however, we concentrate on making Black's model computable. For this purpose we simply specify that (under the distribution associated with E_*)

(b) the value of the underlying instrument at maturity, V_T, is lognormal; in other words, V_T has the form e^X where X is Gaussian.

(c) the mean $E_*[V_T]$ is the forward price of V (for contracts written at time 0, with delivery date T).

We have not specified the variance of $X = \log V_T$; it must be given as data. It is customary to specify the "volatility of the forward price" σ, with the convention that

$$\log V_T \text{ has standard deviation } \sigma\sqrt{T}.$$

Notice that the Gaussian random variable $X = \log V_T$ is fully specified by knowledge of its standard deviation $\sigma\sqrt{T}$ and the mean of its exponential $E_*[e^X]$, since if X has mean m then $E_*[e^X] = \exp(m + \frac{1}{2}\sigma^2 T)$.

Most of the practical examples involve calls or puts. For a call, with payoff $(V_T - K)_+$, hypothesis (b) gives

$$E_*[(V_T - K)_+] = E_*[V]N(d_1) - KN(d_2)$$

where

$$d_1 = \frac{\log(E_*[V_T]/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(E_*[V_T]/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

This is a direct consequence of the lemma we used long ago (in Section 5) to evaluate the Black-Scholes formula. Using hypotheses (a) and (c) we get

$$\text{value of a call} = B(0,T)[F_0 N(d_1) - KN(d_2)]$$

where F_0 is the forward price of V today, for delivery at time T, and

$$d_1 = \frac{\log(F_0/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(F_0/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

A parallel discussion applies for a put.

It is not obvious (at least not to me) that Black's formula is correct in a stochastic interest rate setting. We'll justify it a little later, for options on bonds. But here is a rough, heuristic justification. Since the value of the underlying security is stochastic, we may think of it as having its own lognormal dynamics. If we treat the risk-free rate as being constant then Black's formula can certainly be used. Since the payoff takes place at time T, the only reasonable constant interest rate to use is the one for which $e^{-rT} = B(0,T)$, and this leads to the version of Black's formula given above.

Black's model applied to options on bonds. The following examples is taken from Clewlow and Strickland (section 6.6.1). Let us price a one-year European call option on a discount bond that matures 5 years from now.[1] Assume:

- The current term structure is flat at 5 percent per annum; in other words $B(0,t) = e^{-.05t}$ when t is measured in years.

[1] Edited 12/12/2012.

- The strike of the option is 0.8; in other words the payoff is $(B(1,5)-0.8)_+$ at time $T=1$.
- The forward bond price volatility σ is 10 percent.

Then the forward bond price is $F_0 = B(0,5)/B(0,1) = .8187$ so

$$d_1 = \frac{\log(.8187/.8000) + \frac{1}{2}(0.1)^2(1)}{(0.1)\sqrt{1}} = 0.2814, \quad d_2 = d_1 - \sigma\sqrt{T} = 0.2814 - 0.1\sqrt{1} = .1814$$

and the discount factor for income received at the maturity of the option is $B(0,1) = .9512$. So the value of the call option now, at time 0, is

$$.9512[.8187N(.2814) - .8N(.1814)] = .0404.$$

Black's formula can also be used to value options on coupon-paying bonds; no new principles are involved, but the calculation of the forward price of the bond must take into account the coupons and their payment dates; see Hull's Example 28.1.

One should avoid using the same σ for options with different maturities. And one should never use the same σ for underlyings with different maturities. Here's why: suppose the option has maturity T and the underlying bond has maturity $T' > T$. Then the value V_t of the underlying is known at both $t=0$ (all market data is known at time 0) and at $t=T'$ (all bonds tend to their par values as t approaches maturity). So the variance of V_t vanishes at both $t=0$ and $t=T'$. A common model (if simplified) model says the variance of V_t is $\sigma_0^2 t(T'-t)$ with σ_0 constant, for all $0<t<T'$. In this case the variance of V_T is $\sigma_0^2 T(T'-T)$, in other words $\sigma = \sigma_0\sqrt{T'-T}$. Thus σ depends on the time-to-maturity $T'-T$. In practice σ – or more precisely $\sigma\sqrt{T}$ – is usually inferred from market data.

Black's model applied to caps. A cap provides, at each coupon date of a bond, the difference between the payment associated with a floating rate and that associated with a specified cap rate, if this difference is positive. The ith caplet is associated with the time interval (t_i, t_{i+1}); if $R_i = R(t_i, t_{i+1})$ is the term rate for this interval, R_K is the cap rate, and L is the principal, then the ith caplet pays

$$L \cdot (t_{i+1} - t_i) \cdot (R_i - R_K)_+$$

at time t_{i+1}. Its value according to Black's formula is therefore

$$B(0, t_{i+1}) L \Delta_i t [f_i N(d_1) - R_K N(d_2)].$$

Here $\Delta_i t = t_{i+1} - t_i$; $f_i = f_0(t_i, t_{i+1})$ is the forward term rate for time interval under consideration, defined by

$$\frac{1}{1 + f_i \Delta_i t} = \frac{B(0, t_{i+1})}{B(0, t_i)};$$

and

$$d_1 = \frac{\log(f_i/R_K) + \frac{1}{2}\sigma_i^2 t_i}{\sigma_i\sqrt{t_i}}, \quad d_2 = \frac{\log(f_i/R_K) - \frac{1}{2}\sigma_i^2 t_i}{\sigma_i\sqrt{t_i}} = d_1 - \sigma_i\sqrt{t_i}.$$

The volatilities σ_i must be specified for each i; in practice they are inferred from market data. The value of a cap is obtained by adding the values of its caplets.

A floor is to a cap as a put is to a call: using the same notation as above, the ith floorlet pays

$$L\Delta_i t(R_K - R_i)_+$$

at time t_{i+1}. Its value according to Black's formula is therefore

$$B(0, t_{i+1})L\Delta_i t[R_K N(-d_2) - f_i N(-d_1)]$$

where d_1 and d_2 are as above. The value of a floor is obtained by adding the values of its floorlets.

The market convention is to quote a single volatility for a cap or floor which is then applied to each of the constituent caplets or floorlets – but this is just a convention to make it easy to communicate. To actually price a cap or floor one must evaluate each individual FRA option (caplet or floorlet) at the appropriate volatility, then sum the resulting prices to arrive at the price of the cap or floor. Then one can solve for a single volatility that, applied to each individual FRA option, would give the same price.

Here's an example, taken from Section 28.2 of Hull. Consider a contract that caps the interest on a 3-month, \$10,000 loan one year from now; we suppose the interest is capped at 8% per annum (compounded quarterly). This is a simple caplet, with $t_1 = 1$ year and $t_2 = 1.25$ years. To value it, we need:

- The forward term rate for a 3-month loan starting one year from now; suppose this is 7% per annum (compounded quarterly).
- The discount factor associated to income 15 months from now; suppose this is .9169.
- The volatility of the 3-month forward rate underlying the caplet; suppose this is 0.20.

With this data, we obtain

$$d_1 = \frac{\log(.07/.08) + \frac{1}{2}(0.2)^2(1)}{0.2\sqrt{1}} = -0.5677, \quad d_2 = d_1 - 0.2\sqrt{1} = -0.7677$$

so the value of the caplet is, according to Black's formula,

$$(.9169)(10,000)(1/4)[.07N(-.5677) - .08N(-.7677)] = 5.162 \text{ dollars.}$$

Black's model applied to swaptions. A swaption is an option to enter into a swap at some future date t (the maturity of the option), with a specified fixed rate c and frequency f, which lasts until a specified later time T. (This is evidently a $T - t$-year swap.) For example, if it's an option to pay the fixed rate, and if $c = 5.50\%$, $t = 3$ years and $T = 8$ years, then option can be exercised in 3 years to enter into a 5 year swap with a 5.50% coupon.

To value it, recall that if the payment times are t_j then the value of the swap at time t will be

$$V_{\text{swap}}(t) = L\left[\sum \frac{c}{f}B(t,t_j) + B(t,T) - 1\right],$$

and the par swap rate at time t will be the value of c that makes the right hand side equal to 0:

$$R_{\text{swap}}(t)\sum \frac{1}{f}B(t,t_j) = 1 - B(t,T).$$

Of course $R_{\text{swap}}(t)$ isn't known now, because it dependes on discount rates for lending at time t. But we get the *forward swap rate* F_{swap} by replacing $B(t,t_j)$ above by the forward rate $B(0,t_j)/B(0,t)$: after some arithmetic,

$$F_{\text{swap}}(0)\cdot\sum \frac{1}{f}B(0,t_j) = B(0,t) - B(0,T).$$

If the coupon is set to F_{swap}, then the swap has no value at time 0. The forward swap rate can be calculated at any time $0 \leq \tau \leq t$ of course: arguing as above, it is

$$F_{\text{swap}}(\tau)\cdot\sum \frac{1}{f}B(\tau,t_j) = B(\tau,t) - B(\tau,T)$$

and it agrees with R_{swap} at $\tau = t$.

The swaption will be in the money if the proposed coupon rate c is better than the swap rate R_{swap} when the option matures (time t). For a swap to receive floating rate and pay fixed rate, this occurs if $R_{\text{swap}} > c$. If it is exercised, the value of the swap at exercise is

$$(R_{\text{swap}} - c)\frac{L}{f}\sum B(t,t_j),$$

i.e. the exercised swap has the same value as a stream of payments of $(R_{\text{swap}} - c)\frac{L}{f}$ at each coupon date t_j. Black's formula gives the time-0 value of the jth payment as

$$B(0,t_j)\frac{L}{f}[F_{\text{swap}}N(d_1) - cN(d_2)]$$

where F_{swap} is the forward swap rate and

$$d_1 = \frac{\log(F_{\text{swap}}/c) + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}, \quad d_2 = \frac{\log(F_{\text{swap}}/c) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}.$$

To get the value of the swaption itself we sum over all i:

$$\text{value of swaption} = A[F_{\text{swap}}N(d_1) - cN(d_2)] \quad \text{where } A = \tfrac{L}{f}\textstyle\sum_{i=1}^{N} B(0,t_t).$$

Here σ is the volatility of forward swap rate F_{swap} (which would normally be determined by calibrating the predictions of the model to market prices).

The option to enter into a swap that receives the floating rate and pays the fixed rate uses the call option formula. By entirely similar reasoning, an option to enter into a swap that pays the floating rate and receives the fixed rate uses the put option formula. There is another type of interest rate option that consists of a swap that can be cancelled at a given point in time. The right to cancel a swap to receive the floating rate and pay the fixed rate is equivalent to having the option to enter into a swap paying the floating rate and receiving the fixed rate, thereby offsetting the existing swap. Therefore, the option to cancel a swap receiving floating and paying fixed uses the put option formula. Similarly, an option to cancel a swap paying floating and receiving fixed uses the call option formula.

Options on swaps can be either cash-settlement or settled by delivery. Settlement by delivery involves actually entering into the swap specified. Cash settlement means that the value of entering into this swap at the market rate prevailing at the time of settlement is calculated and then a cash payment is made of this value. Market convention is that caps and floors are always cash-settled.

When we calculate the value of an option on a swap, we are only looking at the value of the fixed rate bond portion of the swap (we have implicitly been assuming that we are always valuing options for dates on which the swap has just made a coupon payment, so that the floating rate bond portion is worth par). Therefore, options on fixed rate bonds can be valued using the exact same formula as options on swaps.

Here's an example, taken from Clewlow and Strickland section 6.6.1. Suppose the yield curve is flat at 5 percent per annum (continuously compounded). Let us price an option that matures in 2 years and gives its holder the right to enter a one-year swap with semiannual payments, receiving floating rate and paying fixed term rate 5 percent per annum. We suppose the volatility of the forward swap rate is 20% per annum.

The first step is to find the forward swap rate F_{swap}. It satisfies

$$F_{\text{swap}}(1/2)\sum_{i=1}^{2} B(0,t_i) = (B(0,t) - B(0,t_2))$$

with $t = 2$, $t_1 = 2.5$, and $t_2 = 3.0$. Since the yield curve is flat at 5% compounded continuously, we have $B(0,t) = e^{-.05t}$ for all t. A bit of arithmetic gives gives $F_{\text{swap}} = .0506$, in other words 5.06%. Now

$$d_1 = \frac{\log(.0506/.0500) + \frac{1}{2}(0.2)^2(2)}{0.2\sqrt{2}} = 0.1587, \quad d_2 = d_1 - 0.2\sqrt{2} = -.0971,$$

and

$$\sum_{i=1}^{2} B(0,t_i)(t_i - t_{i-1}) = \tfrac{1}{2}(e^{-(.05)(2.5)} + e^{-(.05)(3)}) = .8716,$$

so the value of the swaption is

$$.8716L[.0506N(.1587) - .05N(-.0971)] = .0052L$$

where L is the notional principal of the underlying swap.

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When can Black's model be used? Why is it correct? Black's model is widely-used and appropriate for pricing European-style options on interest-based instruments. It has two key advantages: (a) simplicity, and (b) directness. By simplicity I mean not that Black's model is easy to understand, but rather that it requires just one parameter (the volatility) to be inferred from market data. By directness I mean that we model the underlying instrument directly – the basic hypothesis of Black's model is the lognormal character of the underlying.

Black's model *cannot* be used, however, to value American-style options, i.e. options with whose exercise date is not fixed in advance. Many bond options permit early exercise – sometimes American-style (permitting exercise at any time) but more commonly Bermudan (permitting exercise at a list of specified dates, typically coupon dates). Such options are best modelled using a tree (much as we did a few weeks ago for American options on equities). We'll discuss interest rate trees in a moment, but briefly: the tree models the risk-neutral interest-rate process, which can then be used to value bonds of all types and maturities, and American as well as European options on these bonds. Interest rate trees are not "simple" in the sense used above: to get started we must calibrate the entire tree to market data (e.g. the yield curve). And they are not "direct" in the sense used above: we are modeling the risk-neutral interest rate process, not the underlying instrument itself; thus there are two potential sources of modeling error: one in modeling the value of the underlying instrument, the other in modeling how the option's value depends on that of the underlying instrument.

The simplicity and directness of Black's model are also responsible for its disadvantages. Black's model must be used separately for each class of instruments – we cannot use it, for example, to hedge a cap using bonds of various maturities. For consistent pricing and hedging of multiple instruments one must use a more fundamental model such as an interest rate tree.

Now we turn to the question of *why* Black's model is correct. The following explanation involves "change of numeraire". (The word numeraire refers to a choice of units.)

Up to now our numeraire has been cash (dollars). Its growth as a function of time is described by the money-market account introduced in Section 7. The money-market account has balance is $A(0) = 1$ initially, and its balance evolves in time by $A_{\text{next}} = e^{r\delta t} A_{\text{now}}$. We are accustomed to finding the value f of a tradeable instrument (such as an option) by working backward in the tree using the risk-neutral probabilities. At each step this amounts to

$$f_{\text{now}} = e^{-r\delta t}[q f_{\text{up}} + (1-q) f_{\text{down}}]$$

where q and $1-q$ are the risk-neutral probabilities of the up and down states. As we noted in Section 7, this can be expressed as

$$f_{\text{now}}/A_{\text{now}} = E_{\text{RN}}[f_{\text{next}}/A_{\text{next}}],$$

and it can be iterated in time to give

$$f(t)/A(t) = E_{\text{RN}}[f(t')/A(t')] \quad \text{for } t < t'.$$

This is captured by the statement that "$f(t)/A(t)$ is a martingale relative to the risk-neutral probabilities."

But sometimes the money-market account is not the convenient comparison. In fact we may use *any* tradeable security as the numeraire – though when we do so we must also change the probabilities. Indeed, for any tradeable security g there is a choice of probabilities on the tree such that

$$\frac{f_{\text{now}}}{g_{\text{now}}} = \left[q_* \frac{f_{\text{up}}}{g_{\text{up}}} + (1 - q_*) \frac{f_{\text{down}}}{g_{\text{down}}} \right].$$

This is an easy consequence of the two relations

$$f_{\text{now}} = e^{-r\delta t}[q f_{\text{up}} + (1-q) f_{\text{down}}] \quad \text{and} \quad g_{\text{now}} = e^{-r\delta t}[q g_{\text{up}} + (1-q) g_{\text{down}}],$$

which hold (using the risk-neutral q) since both f and g are tradeable. A little algebra shows that these relations imply the preceding formula with

$$q_* = \frac{q g_{\text{up}}}{q g_{\text{up}} + (1-q) g_{\text{down}}}.$$

(The value of q_* now varies from one binomial subtree to another, even if q was uniform throughout the tree.) Writing E_* for the expectation operator with weight q_*, we have defined q_* so that

$$f_{\text{now}}/g_{\text{now}} = E_*[f_{\text{next}}/g_{\text{next}}].$$

Iterating this relation gives (as in the risk-neutral case)

$$f(t)/g(t) = E_*[f(t')/g(t')] \quad \text{for } t < t';$$

in other words "$f(t)/g(t)$ is a martingale relative to the probability associated with E_*." In particular

$$f(0)/g(0) = E_*[f(T)/g(T)]$$

where T is the maturity of an option we may wish to price.

Let us apply this result to explain why Black's formula is valid.[2] For simplicity we focus on the special case of a call option on a zero coupon bond. Fixing the notation: let T be the maturity of the option, let K be its strike, and let $T' > T$ be the maturity of the bond. So the holder of the option has the right to purchase, at time T, paying K, a zero-coupon bond worth one dollar at its maturity T'. The option's payoff (at time T) is therefore $(B(T,T') - K)_+$. To value it using Black's formula, we choose a zero-coupon bond with maturity T as the numeraire; in other words, we use the discussion above with

$$g(t) = B(t,T).$$

Since $g(T) = 1$ this choice gives

$$f(0) = g(0)E_*[f(T)] = B(0,T)E_*[f(T)]$$

whenever f is the price of a tradeable. We apply this formula twice:

[2] The following discussion was rewritten on 11/28/2012.

(a) Let's apply it to the zero-coupon bond maturing at T' (which is the our option's "underlying"). Substituting $f(t) = B(t, T')$ in the preceding formula gives

$$B(0, T') = B(0, T)E_*[B(T, T')].$$

Thus $E_*[B(T, T')] = B(0, T')/B(0, T)$, which we recognize as the forward price of the underlying.

(b) Now let's apply it to the option. Writing $V(t)$ for the value of the option at time t, we have

$$V(0) = B(0, T)E_*[(B(T, T') - K)_+]$$

since at time T the value of the option is precisely its payoff. Black's formula (using the variant for a call) follows immediately, provided that $B(T, T')$ is lognormal (under the distribution associated with E_*).

Why should $B(T, T')$ be lognormal? This is true for many widely-used (continuous-time) interest rate models. But defining and proving results about interest rate models in a continuous-time setting requires methods from stochastic calculus, so it lies beyond the scope of this class. (Well, we'll get our feet wet next week, by discussing the Vasicek short-rate model.)

We have focused on a call option on a zero-coupon bond. The pricing of a put on a zero-coupon bond is similar. In fact, $g(t) = B(t, T)$ is the right choice for any instrument whose payoff involves a cash flow paid at the maturity time T. However, the justification of Black's formula for caplets, floorlets, or swaptions is a bit different, because their cash flows occur at times other than the maturity of the option. (See Hull for the justification in these cases.)

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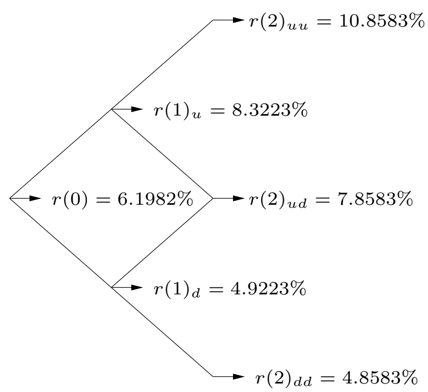
Interest rate trees and American style interest rate options. Thus far we have only discussed European-style interest rate options – ones where there is a single option exercise date. But there are interest rate derivatives which permit multiple exercise dates: a particularly popular product in the US Dollar interest rate market is the Bermudean swaption, an option to enter into a particular swap on any of a series of payment dates for the swap. For example, the underlying swap might be one to pay 6 month LIBOR and receive a fixed coupon of 6% payable semiannually through Sept. 30, 2017 (the "maturity date" of the swap). The Bermudean option could be exercisable every 6 months starting Sept. 30, 2008 and ending Sept. 30, 2013 – the maturity date of the swap stays fixed at Sept. 30, 2017.

Valuation of American style interest rate options is almost always performed using a binomial or trinomial tree. This is conceptually the same as what we did for American-style options on forwards. However, there is an important difference. In pricing options on equities, we specified a tree for market price of a forward, which had to be a martingale under the risk-neutral measure. (We used this to determine the risk-neutral measure.) For pricing interest-based instruments, we will specify a tree for the *short-term interest rate under the risk-neutral measure.* The short-term interest rate is not the price of an asset, so there is no reason for it to be a martingale.

[Let's pause for a brief digression. You might wonder why we don't build a tree based on the price of an asset, an interest rate forward or a swap, rather than on an interest rate. There are basically two reasons: (1) if we made the convenient assumption that the price of the asset follows a lognormal process, it might become so high on some nodes that it would imply a negative interest rate and negative interest rates only occur in very extraordinary circumstances; (2) as a swap gets close to maturity, its price must go towards par (its duration is getting shorter and shorter) so it certainly can't be described by a martingale).]

How does an interest rate tree work? The basic idea is shown in the figure: each node of the tree is assigned a risk-free rate, different from node to node; it is the one-period risk-free rate for the binomial subtree just to the right of that node.



What probabilities should we assign to the branches? It might seem natural to start by figuring out what the subjective probabilities are. But why bother? All we really need for option pricing are the risk-neutral probabilities. When we discussed equities we used the volatility and drift of the forward price to establish a tree, then used its nodes to find the risk-neutral probabilities. But recall that when we considered the continuum limit $\delta t \to 0$, the risk-neutral probabilities were very close to 1/2. For an interest rate tree we have the right to choose the branching probabilities as we please; the usual practice is to make them exactly 1/2.

But we don't have complete freedom. The discount rates $B(0, T)$ are known at time 0 for all maturities T. To be useful, our interest rate tree must agree with this market data. In other words, it must be *calibrated* to the present term structure in the marketplace. In summary: for interest rate trees we

- restrict attention to the risk-neutral interest rate process.
- assume the risk-neutral probability is $q = 1/2$ at each branch, and

- choose the interest rates at the various nodes so that the long-term interest rates associated with the tree match those observed in the marketplace.

The last bullet – calibration of the tree to market information – is the most subtle one; we'll discuss it only briefly. (For more information about calibration in the context of the Hull-White model, see section 30.7 of Hull.)

First let's just be sure we understand how our tree determines long-term interest rates. As an example let's determine $B(0,3)$, the value at time 0 of a dollar received at time 3, for the tree shown in the figure. (Put differently: $B(0,3)$ is the price at time 0 of a zero-coupon bond which matures at time 3.) We take the convention that $\delta t = 1$ for simplicity.

Consider first time period 2. The value at time 2 of a dollar received at time 3 is $B(2,3)$; it has a different value at each time-2 node. These values are computed from the fact that

$$B(2,3) = e^{-r\delta t}[\tfrac{1}{2}B(3,3)_{\text{up}} + \tfrac{1}{2}B(3,3)_{\text{down}}] = e^{-r\delta t}$$

since $B(3,3) = 1$ in every state, by definition. Thus

$$B(2,3) = \begin{cases} e^{-r(2)_{uu}} = .897104 & \text{at node } uu \\ e^{-r(2)_{ud}} = .924425 & \text{at node } ud \\ e^{-r(2)_{dd}} = .952578 & \text{at node } dd. \end{cases}$$

Now we have the information needed to compute $B(1,3)$, the value at time 1 of a dollar received at time 3. Applying the rule

$$B(1,3) = e^{-r\delta t}[\tfrac{1}{2}B(2,3)_{\text{up}} + \tfrac{1}{2}B(2,3)_{\text{down}}]$$

at each node gives

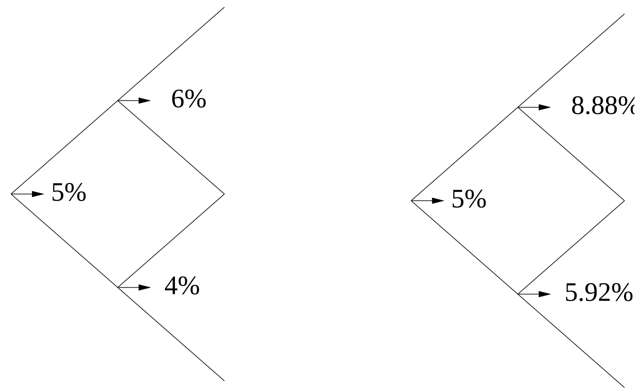
$$B(1,3) = \begin{cases} e^{-r(1)_u}(\tfrac{1}{2}\cdot .897104 + \tfrac{1}{2}\cdot .924425) = .838036 & \text{at node } u \\ e^{-r(1)_d}(\tfrac{1}{2}\cdot .924425 + \tfrac{1}{2}\cdot .952578) = .893424 & \text{at node } d. \end{cases}$$

Finally we compute $B(0,3)$ by applying the same rule:

$$\begin{aligned} B(0,3) &= e^{-r\delta t}[\tfrac{1}{2}B(1,3)_{\text{up}} + \tfrac{1}{2}B(1,3)_{\text{down}}] \\ &= e^{-r(0)}[\tfrac{1}{2}\cdot .838036 + \tfrac{1}{2}\cdot .893424] = .8137. \end{aligned}$$

Valuing an option using an interest rate tree is easy: just work backward (the option is a tradeable). Hedging is easy too: each binomial submarket is complete, so a risky instrument can be hedged using any pair of interest-based insruments (for example, two distinct zero-coupon bonds).

Here is a toy (two-period) example to communicate the idea of calibration. Recall that if $A(t)$ is the balance of a money-market fund with value 1 at time 0, then the value of a European option with maturity T is $E_{\text{RN}}[\text{payoff}/A(T)]$. For a zero-coupon bond with maturity T, the payoff is 1, so the value of the option is $E_[1/A(T)]$. Suppose that the present marketplace yields are $y(0,1) = 5\%$ and $y(0,2) = 6\%$, so $B(0,1) = e^{-.05} = .9512$



and $B(0,2) = e^{-.06*2} = .8869$. Suppose you have guessed that the interest rate tree (with branching probabilities 1/2) has nodal yields 5% initially, branching to 4% and 6% after one year as shown in the left side of the next figure. Such a tree gives the price of the discount bound as

$$B(0,2) = \tfrac{1}{2}\left[\frac{1}{1.06 * 1.05} + \frac{1}{1.04 * 1.05}\right] = \tfrac{1}{2}(.8985 + .9158) = .9071$$

which is far off from the observed value .8869. There is a systematic algorithm for correcting this (see e.g. sections 8.1-8.4 of Clewlow and Strickland). The output of such an algorithm might for example be the tree shown on the right hand side of the figure. In fact, it gives the proper value for $B(0,2)$, since its prediction is

$$B(0,2) = \tfrac{1}{2}\left[\frac{1}{1.088 * 1.05} + \frac{1}{1.0592 * 1.05}\right] = \tfrac{1}{2}(.874 + .8992) = .8869.$$

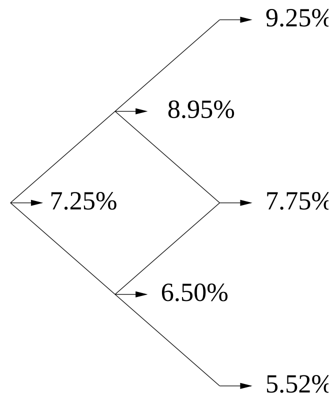
You may be disturbed by this example – how could a 5% interest rate evolve to either 8.88% or 5.92%? Not only is the average of these rates 7.40%, not equal to 5%, but even the lower of the two rates is greater than 5%. Don't we have an arbitrage? No we don't, because an interest rate is not the price of a tradable asset.

In fact, the rates in a tree are not related to one another by any theory. The 5% rate was the rate that applied to the first year; the 8.88% and 5.92% rates where the ones that applied in the second year. The observable quantity that's most closely related to 8.88% and 5.92% to is the current 1 year forward rate, which is 7% since $e^{-.06*2} = e^{-.05*2}e^{-.07*2}$.

A key issue in building interest rate trees is how much *mean reversion* to build into the tree. Mean reversion refers to a negative correlation between rate levels in one period and rate levels in the immediately succeeding period. When you have mean reversion, the expected rate for the following period will be lower than the current rate for higher rate nodes and higher than the current rate for lower rate nodes. Mean reversion is an issue that does not arise for trees that are being used to model tradable assets; tradable asset prices are martingales (more precisely: their present-valued prices are martingales), so that every node must have the current price equal to the expectation of the price at the following period.

This is inconsistent with mean reversion. But interest rates are not tradable assets. One reason mean reversion is an economically reasonable assumption for interest rates is that the central bank tends to act as a counterbalancing force in the economy and pull rates back towards target levels. The reasonableness of mean reversion assumptions can be seen from historical data, by noting that the volatility of forward rates with later starting dates is lower than those with shorter starting dates.

Any interest-based instrument can be priced on a tree. To price a swap, you'll need access to more than a single year's worth of forward prices. But you can easily access this by calculations on the tree. For example, let's say you are at a node where you need to calculate the price of an annual pay swap with 3 years remaining that pays a fixed coupon of 6%. Let's say the 1 year rates at your node and its immediate branches look like the following figure: The one year discount factor is $e^{-.0725} = .9301$. The two year discount factor is the



average of $e^{-.0725}e^{-.0895} = .8504$ and $e^{-.0725}e^{-.065} = .8715$, which is .8610. The three year discount factor is the average of $e^{-.0725}e^{-.0895}e^{-.0925} = .7753$, $e^{-.0725}e^{-.0895}e^{-.0775} = .7870$, $e^{-.0725}e^{-.065}e^{-.0775} = .8065$, and $e^{-.0725}e^{-.065}e^{-.0525} = .8270$, which is .8464. So the value of the swap is $100 * .8464 + 6(.9301 + .8610 + .8464) - 100 = .4650$.