The relevant parts of Hull are:

- chapter 8, on structured credit products + the recent financial crisis
- sections 23.9 - 23.10, on Gaussian copula approach to default correlations
- section 24.3 about CDS indexes
- sections 24.8 - 24.10 on CDO's

In all we focused on bonds issued by a specific corporation, or CDS on a single entity. Therefore only the prob of default by this entity was relevant.

In multiname setting, correlation between defaults is also important.

**Example:** Suppose you hold bonds issued by 10 companies, and each has a default prob of 1% in 1st yr. If independent, then

\[
\text{prob of exactly } k \text{ defaults in 1st yr } = \binom{10}{k} (0.01)^k (0.99)^{10-k}
\]

so, for example, prob that all default
is \((0.1)^{10} = 10^{-20}\). (Exceedingly small!)

If fully correlated, then either there’s no default or they all default; in this case

\[
\text{prob that all default in 1st year} = 0.01 \quad \text{(not so small!)}
\]

Why do we care?

1. A bank with many credit exposures must estimate its “value at risk.” This amounts to estimating the likelihood of a very large loss (due to defaults) within a specific time period.

2. An investor may seek to protect itself in a general, diversified way from deterioration of credit environment; or to profit by selling such protection. The standard tool is a basket CDS or index CDS contract. Index CDS works like this:

- There’s a specific list of companies (eg. CDX NA IG corresponds to list of 125 investment grade companies in North America)
contract behaves like an (equally-weighted) CDO on each company. For example:

1st default $\rightarrow$ seller of protection pays $L(1-R) \cdot \frac{1}{125}$ to purchaser, and "spread payments" are reduced to $\frac{124}{125}$ of initial amount.

2nd default $\rightarrow$ seller pays another $L(1-R) \cdot \frac{1}{125}$ to purchaser, and "spread payments" are reset to $\frac{123}{125}$ of initial amount.

To value this, we need expected number of defaults at each payment date.

Note: recent huge trading loss by J.P. Morgan Chase involved contracts of this type. (Why? The big loss? Huge trades $\rightarrow$ they moved the market $\rightarrow$ price paid was artificially high $\rightarrow$ after a while they had a huge loss.)

(3) Before 2007, CDO's (collateralized debt obligations) backed by mortgages were widely
created + traded. Their purpose: to create high-quality bonds out of lower-quality loans (e.g. subprime). Also used for credit card debt, etc. Much less-used since 2008

**How it works:**

\[
\begin{align*}
\text{collection of} & \quad \rightarrow \quad \text{senior tranche,} \\
\text{lower-quality} & \quad \rightarrow \quad \text{principal 80%} \\
\text{loans} & \quad \rightarrow \quad \text{pays Libor + 60 basis pts} \\
\text{total prn = 100M} & \quad \rightarrow \quad \text{mezzanine tranche} \\
& \quad \rightarrow \quad \text{principal 15%} \\
& \quad \rightarrow \quad \text{pays Libor + 250 basis pts} \\
& \quad \rightarrow \quad \text{Equity tranche} \\
& \quad \rightarrow \quad \text{principal 5%} \\
& \quad \rightarrow \quad \text{pays Libor + 2000 basis pts}
\end{align*}
\]

(Hull, §8.1) - here

- equity tranche bears 1st 5% of defaults (after which it's dead);
- mezzanine tranche bears next 15% of defaults (then it too is dead);
- senior tranche starts seeing losses if defaults exceed 20%.

In practice there could be many more tranches
Typically (based on hypothesis of not too large correlation) senior tranche would be high quality even if underlying instruments aren't. So it's a way to draw capital that otherwise wouldn't have been available.

Worked badly for many reasons:
- correlations in 2007-2008 were much greater than was assumed
- creator of underlying bonds has no incentive to be honest or maintain quality

4 Also pre-2007: people also created synthetic CDS's, which were essentially tranched versions of CDS's on a portfolio.

- equity tranche earns higher spread but bears full face of initial defaults (up to 5% say)
- mezzanine tranche earns interme spread + bears face of defaults beyond 5% up to 15% (say)
- senior tranche gets smallest spread + bears face of defaults only after 20%
Why? Basically same reasons as regular CDS's in a portfolio. But much more complex to price or hedge.

Why were they popular? Mostly as a speculative tool, I think.

Stepping back: tranching instruments are no longer so popular, but correlations between defaults are still important for basket or index CDS, or estimating VAR on a debt portfolio.

Basic tool: Gaussian copula permits disentangle correlation between two unrelated, non-Gaussian RV's.

Define \( x_1(t_1) \) by
\[
N(x_1(t_1)) = \text{prob that RV #1 \leq t_1}
\]
+ similarly \( x_2(t) \) (using RV\#2). Now assume
\( x_1 + x_2 \) are bivariate normal with corr \( p \).

Let's apply this viewpoint to estimate default risk involving a pool of entities, assuming for simplicity:

- homogeneity (all entities are equivalent)
- a one-factor model (to be explained below)

(think of the common factor as the "state of the economy")

- the pool is large (permitting us to use law of large numbers)

Goal is an expression for fraction of the pool that has defaulted by time \( t \) (or just the expected number of defaults by time \( t \)), given inputs

- prob that a single entity defaults by time \( t \) is given, say \( D = D(t) \)
- correlation \( p \) (and Gaussian copula).

Let \( X_j \) be the standard Gaussian assoc to the jth entity. Then
\[ x_j = \frac{1}{2} F + \sqrt{1 - \frac{1}{2}} Z_j \quad \text{(the one-factor hypothesis)} \]

where \( F + Z \) are independent standard Gaussians.

(Note that \( E\xi_j = 0, \ E\xi_j^2 = 1, \ E(\xi_j \xi_k) = p \).

So: if value of \( F \) is fixed, then

name \( j \) defaults \( \iff x_j \leq x^{-1}(D) \)

\[ \iff Z_j < \frac{x^{-1}(D) - \sqrt{p} F}{\sqrt{1-p}} \]

\[ \iff \text{prob of default is } N\left( \frac{x^{-1}(D) - \sqrt{p} F}{\sqrt{1-p}} \right) \]

To find prob that any indiv. credit defaults, we're thus left to do a numerical integration

\[ \text{prob} = \int N\left( \frac{x^{-1}(D) - \sqrt{p} F}{\sqrt{1-p}} \right) \cdot \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds \]

If pool is large, it's a decent approxm to identify this with the fraction of the pool that has defaulted by \( x^{-1}(D) \).
Uses of this:

a) for estimating VAR:

What is the likelihood that 10% of the pool defaults by time $t$?

\[ \left( \text{under large-pool approx.} \right) \]

What is the likelihood that

\[ N \left( \frac{N^{-1}(D) - \sqrt{p} \cdot F}{\sqrt{p}} \right) > 0.1 \]

\[ \uparrow \]

What is the likelihood that a Gaussian $F$ has

\[ F < \frac{N^{-1}(D) - \sqrt{p} \cdot N^{-1}(0.1)}{\sqrt{p}} \]

b) for pricing a CDS on an index:

Protection purchaser pays:

\[ L \frac{d}{dt} \left[ \frac{1}{2} \cdot \mathbb{E} \left[ \text{fraction of names still alive at } t_i \right] \right] \text{ at } t_i \]

and receives

\[ L \left( 1 - R \right) \left( \mathbb{E} \left[ \text{fraction alive at } t_{i-1} \right] - \mathbb{E} \left[ \text{fraction alive at } t_i \right] \right) \]
due to defaults, plus "accrued interest"

$$L = \frac{\bar{\delta}}{2\delta} \left( E \left[ \text{fraction alive} \right]_{\text{at } t_{j-1}} - E \left[ \text{fraction alive} \right]_{\text{at } t_j} \right)$$

(at time \((t_{j-1} + t_j)/2\), by the usual formula).

Note that calcs of \(E \left[ \text{fraction alive at } t_j \right]\) are all "the same" (as explained above), except for use of different values of \(D = \text{prob that any single name has defaulted by time } t\).

c) valuation of a tranched CDO is only a little different. Let

$$\theta \left( F \right) = N \left( \frac{N^{-1}(D^\text{+}) - \sqrt{p} F}{\sqrt{1-p}} \right)$$

= fraction of defaults by time \(t\),
given \(F\) (under large-pool approx).

Then eq. for mezzanine tranche in our example, protection purchaser's payment at time at \(t_j\) is

$$L = \frac{\bar{\delta}}{2\delta} g \left( \theta \right)_{t_j}$$

\[\text{10%} \quad \text{g(θ)} \quad \begin{array}{c}
\text{5%} \\
\text{15%}
\end{array}\]
which gets valued as

\[ L \frac{d}{dt} \mathbb{E}[g(\theta_t)] B(0,t) \]

Gaussian expectation, since \( \theta_t \) is a r.v. of the Gaussian RV \( F \)

and protection purchaser receives (at \( t_{i-1} + \frac{t_i - t_{i-1}}{2} \))

\[ L (1-\kappa) + L \frac{d}{dt} \times g(\theta_{t_{i-1}}) - g(\theta_{t_i}) \]

valued as in

\[ \mathbb{E}[g(\theta_{t_{i-1}})] - \mathbb{E}[g(\theta_{t_i})] \]

(were expectations are to Gaussian \( F \)).

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For all these calims, choice of \( g \) is crucial. Lesson of 1999-2000 blowup and also 2007-8 crisis; inferring \( g \) from market data is dangerous (the market could have errors, beliefs, or be skewed by supply/demand issues, for example.)