Section 11 - picking up where the typed notes leave off - Derivatives Securities, Fall 2012

Pricing corporate bonds + credit default swaps

Must use defaultable discount rate

\[ \tilde{B}(0,T) = \text{value today of a note worth } \$1 \text{ at the } T \text{ provided issuer hasn't defaulted by then} \]

Modeling hypothesis:

\[ \tilde{B}(0,T) = \frac{S_{T}}{\tilde{B}(0,T)} \]

probability * risk-free discount rate of survival to the T

Note: since \( \tilde{B}(0,T) = E_{T_0} \left[ \text{discounted payoff} \right] \)

we see

a) that \( S_{T} = \text{risk-neutral prob of survival to time } T \)

b) essence of the "modeling hypothesis" is that default is independent of changes in interest rates
Evidently, modeling hypothesis leaves us with task of extracting $s_t$ from market data.

A little more language:

\[ d_i = \text{prob of default between } t_{i-1} + t_i \rightarrow \text{cumulative survival up to } t_i \]

\[ t_0 = 0 \quad t_1 \quad t_2 \ldots \quad t_N. \]

so \[ S_0 = 1 \rightarrow S_{i+1} = S_i (1 - d_{i+1}) \]

(Thus \( d_i \) = prob of default in \((t_{i-1}, t_i)\), \( S_i = (1 - d_i) \) is prob of survival to \( t_i \), \( S_i d_{i+1} = \text{prob of default in } (t_i, t_{i+1}) \), \( S_2 = S_1 - S_1 d_2 = \text{prob of survival to } t_2 \), etc.)

New twist re: risk-free bonds: a bond’s coupon payments cease to happen after default \( d \) (creditors cannot claim these in bankruptcy court) but some part of principal will be recovered (after liquidation or other bankruptcy settlement).

Value of a corporate bond: assume
principal $L$, fixed rate $c$, payment frequency $f$ (times/year), recovery rate $R$, payment dates $t_i$ (maturity $t_N$).

$$\Rightarrow \text{Value} = \left[ \sum_{i=1}^{N} \frac{c}{f} S_i B(0, t_i) + S_{t_N} B(0, t_N) \right] L - \sum_{i=1}^{N} S_{i-1} d_i B(0, t_i^*) R L$$

using notation $t_i^* = \frac{t_i + t_{i+1}}{2}$.

Term involving $R$ is the "best estimate" for present value of the (partial) recovery of principal, if default occurs. Why $t_i^*$? Well, given default is presumably equally likely at any time in $(t_{i-1}, t_i)$.

Terminology: par coupon rate = the value of $c$ that makes the bond's value $= L$.

Value of CDS: only slightly different.

Assume $L$, $f$, $R$, $t_i$, $t_N$ are as above.

Let fixed payments be at rate $s$ (the "spread"). Then
Value of fixed payments (paid by buyer of protection) is

\[ V_{\text{fix}} = L \sum_{i=1}^{N} \frac{1}{\delta} S_i B(0, t_i) \]

If there's a default, seller of protection pays buyer \( L(1-R) \) at the time of default (assuming, say, cash settlement) and receives accrued interest owed up to moment of default. So

\[ V_{\text{float}} = L \sum_{i=1}^{N} \frac{S_i - d_i B(0, t_i^*)}{(1-R) - \frac{\delta}{\delta}} \]

estimate of accrued interest

Present value of swap is \( V_{\text{fix}} - V_{\text{float}} \) (This is the value to the seller, or provider, of protection.)

Par OIS spread is the value of \( s \) that makes \( V_{\text{fix}} - V_{\text{float}} = 0 \) (Note that \( V_{\text{fix}} - V_{\text{float}} \) has the form \( as - b \), so \( s = \frac{b}{a} \)).

See Hull 224.2 for a numerical example.
using precisely this framework.

Variations are possible depending on details of CDS contract (eg it might provide a lump-sum payment instead of delivery of a bond)

Note that all the “default probabilities” above (typically extracted from market prices of bonds or swaps) are risk-neutral probs.

They are not the same as real-world probs. Usually:

real-world default prob < RN default prob

(see 8.23.5 of Hull) i.e. in other words, defaultable bonds are cheaper than the historical default rates would suggest.

Possible reasons (somewhat speculative):

• “flight to quality” clearly had an effect during 2007-2008 crises

• real assc under bonds is back to diversity away; also, bond defaults
are correlated to rest of economy

=> can't be diversified away => by CAPM, investors should be compensated for assuming such risk.

So far, we left default probs to be extracted from bond or CDS prices.

But why not try to get them from stock prices?

Reasons to try:

a) stock prices are clearly related
   (stock becomes worthless upon default)

b) stock prices are liquid and very visible

c) some instruments even combine elements of stock and bond (eg convertible bonds)

First attempt to do this was by Black + Scholes (1973) + Merton (1974) - now known as "Merton's..."
model. A related idea is in widespread current use: the CreditGrades model, developed about 2002 by Deutsche Bank, Goldman Sachs, J P Morgan, + Real Metrics. Let's discuss both.

The Merton Model (Merton, J Financ 29, 1974, 449-470) views company's equity as an option on its assets. Let:

\[ V_0 = \text{value/share of company's assets today} \]
\[ V_T = \text{""""""""""""""portion"" at time } T \]

known →
\[ E_0 = \text{value/share of company's stock today} \]
\[ E_T = \text{""""""""""""""portion"" at time } T \]

known →
\[ D = \text{debt/share the company owes (assumed constant)} \]
\[ \sigma_V = \text{volatility of } V \text{ (assumed constant)} \]
\[ \sigma_E = \text{volatility of stock price} \]

Basic hypothesis:
\[ E_T = (V_T - D)_+ \]

since if \( V_T - D < 0 \) the company is bankrupt at the \( T \), else value of equity = \( V_T - D \).
Consequences, by Black-Scholes:

a) \( E_0 = V_0 \cdot N(d_1) - D e^{-rT} N(d_2) \)

where \( d_1 = \frac{\ln(V_0/D) + (r + \frac{1}{2} \sigma^2)}{\sigma \sqrt{T}} \)
\( d_2 = d_1 - \sigma \sqrt{T} \); also

b) \( \text{The prob of default} = 1 - N(d_2) \).

Need inputs \( \sigma \) and \( V_0 \).

- How to get \( \sigma \)? By Itô,

\[ E(t) = E(V(t), t) \quad \text{given by Black-Scholes formula for a call option} \]

\[ dE = \frac{\delta E}{\delta V} dV + (\text{stuff}) dt \]

\[ \Rightarrow \sigma E = \frac{\delta E}{\delta V} \sigma V \quad \text{by comparing coeffts of } "dW" \text{ on both sides of preceding SDE} \]

Also: \( \frac{\delta E}{\delta V} = N(d_1) \) from our discus of the ' Greeks' and \( \sigma E \) is visible in market as weighted volatility of options.

So

\[ \sigma E E_0 = \frac{\delta E}{\delta V} \sigma V_0 = N(d_1) \sigma V_0 \]

is a nonlinear fit relating \( V_0 \) and \( \sigma \) to market observables.
Stepping back: we now have two nonlinear terms in $V_0$ and $\sigma_v$:

\[
E_0 = V_0 \text{N}(d_1) - D e^{-rt} \text{N}(d_2) \\
\sigma_v E_0 = \text{N}(d_1) \sigma_v V_0
\]

which we can expect to solve (by Newton's method), given observed values of $E_0$, $\sigma_v$, and $D$.

How well does this work? Well, it's not reasonable to take $\text{N}(d_2)$ as the "exact" RN default prob. (For example: predicted probs of default in 1st year are much too low.) But the ordering of default probs obtained this way is about right (as the model is applied to different firms).

Easy to criticize: e.g. doesn't distinguish between default at $T$ vs. default before $T$. Also, why should $V(t)$ be log-normal? (Default often occurs due to surprises; why shouldn't $V(t)$ have jumps?)

The CreditGrades model strives to keep the simplicity of Merton's model but make it more predictive. Described by "CreditGrades Technical Document," C. Finger et al., Riskmetrics 2002 (easy to find via google). Starts like Merton's model, however:
\[ D V = \mathcal{G} V \text{ dw} \quad (\text{ie assume drift } = 0) \]

- View default as occurring when \( V(t) \) hits a barrier.

\[ V(t) = V_0 + \mathcal{G} t + \sigma \sqrt{t} W(t) \]

- Heston took barrier \( D \), here we take barrier to be random (ie not entirely known).
  
  \[ \text{barrier} = \mathcal{F} D e^{\gamma Z - \frac{1}{2} \gamma^2} \quad Z = \text{standard Gaussian} \]

  where we view
  
  \[ \mathcal{F} = \mathcal{F} e^{\gamma Z - \frac{1}{2} \gamma^2} \]

  as the \underline{recovery rate} (so \( \mathcal{F} \) and \( \gamma \) are obtainable from market data).

2\text{nd} bullet says

\[ \text{default time} = \inf \text{ such that } V_0 e^{\mathcal{G} W(t) - \sigma^2 t / 2} \text{ crosses barrier} \]
\[ X_t = \sigma_w t - \lambda t - \frac{\sigma_w^2}{2} - \frac{\lambda^2}{2}. \]

There are formulas for such things...

We still have to relate \( V_0 \) and \( \sigma_v \) to market observables; \( V_0 = E_o + \overline{D} \), but what about \( \sigma_v \)?

Recall from any text on Heisenberg that

\[ E \sigma_E = \sigma_v V \frac{\partial E}{\partial V} \]

but we no longer have such simple dependence of \( E_m V \). Argue instead as follows:

Define "distance to default" \( \gamma \) by

\[ \gamma = \frac{1}{\sigma_v} \log \left( \frac{V}{LD} \right), \]

\[ = \frac{V}{\sigma_E} \frac{\partial E}{\partial V} \log \left( \frac{V}{LD} \right) \]

and assume

\[ \gamma = \frac{E + LD}{\sigma_E E} \log \left( \frac{E + LD}{LD} \right) \]
(why? well, it has plausible behavior in the limits $E \ll LD$ and $E \gg LD$; see 3.2.2 of the CreditGrades technical document)

So combining the 1st & last exprs, we get

$$\frac{1}{\sigma_V} = \frac{E_0 + \bar{LD}}{\sigma_E E} \quad \text{i.e.} \quad \sigma_V^2 = \frac{\sigma_E^2 E_0}{E_0 + \bar{LD}}$$

(RHS is known!)

For some examples & discussion, see e.g.
"An empirical implementation of CreditGrades ",
Journal of Credit Risk 6 (2010) 89–98, by
Andy Tse-Yuh Yeh [apparently done as a HS project for the Harvard School's MFE program]
