

Derivative Securities – Fall 2012 – Section 1

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Forwards, puts, calls, and other contingent claims. This section discusses the most basic examples of contingent claims, and explains how considerations of arbitrage determine or restrict their prices. In Hull's 8th edition, this material is distributed between Chapters 1, 2, 3, 5, 9, 10, and 11 (which also provide valuable information not discussed in the notes, e.g. about how various markets work). For a more concise, less distributed treatment see also Chapters 2 and 3 of Jarrow and Turnbull. In Joshi, the corresponding discussion is in Chapters 1 & 2 (but its scope is much narrower). We concentrate for simplicity on European options rather than American ones, on forwards rather than futures, and on deterministic rather than stochastic interest rates.

The most basic instruments:

Forward contract with maturity T and delivery price K.

buy a forward \leftrightarrow hold a long forward
 \leftrightarrow holder is obliged to buy the
underlying asset at price K on date T.

European call option with maturity T and strike price K.

buy a call \leftrightarrow hold a long call
 \leftrightarrow holder is entitled to buy the
underlying asset at price K on date T.

European put option with maturity T and strike price K.

buy a put \leftrightarrow hold a long put
 \leftrightarrow holder is entitled to sell the
underlying asset at price K on date T.

These are *contingent claims*, i.e. their value at maturity is not known in advance. Payoff formulas and diagrams (value at maturity, as a function of S_T =value of the underlying) are shown in the Figure.

Any long position has a corresponding (opposite) *short* position:

Buyer of a claim has a long position \leftrightarrow seller has a short position.

Payoff diagram of short position = negative of payoff diagram of long position.

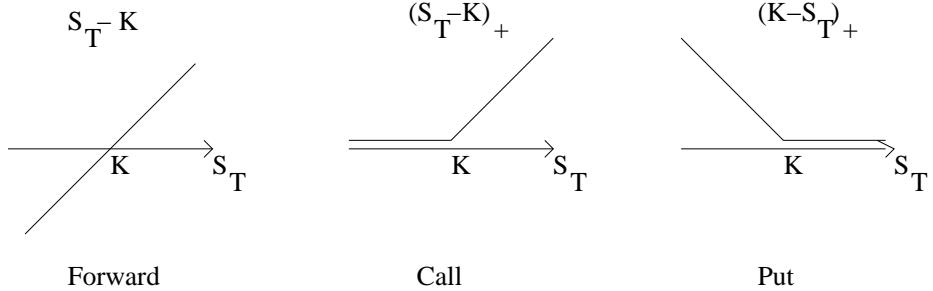


Figure 1: Payoffs of forward, call, and put options.

An *American* option differs from its European sibling by allowing early exercise. For example: the holder of an American call with strike K and maturity T has the right to purchase the underlying for price K at any time $0 \leq t \leq T$. A discussion of American options must deal with two more-or-less independent issues: the unknown future value of the underlying, and the optimal choice of the exercise time. By focusing initially on European options we'll develop an understanding of the first issue before addressing the second.

Why do people buy and sell contingent claims? Briefly, to *hedge* or to *speculate*. Examples of hedging:

- A US airline has a contract to buy a European-made airplane for a price fixed in Euros, payable one year from now. By going long on a forward contract for Euros (payable in dollars) it can eliminate its foreign currency risk.
- The holder of a forward contract has unlimited downside risk. Holding a call limits the downside risk (but buying a call with strike K costs more than buying the forward with delivery price K). Holding one long call and one short call costs less, but gives up some of the upside benefit:

$$(S_T - K_1)_+ - (S_T - K_2)_+ \quad K_1 < K_2$$

This is known as a “bull spread”. (See the figure.)

Options are also frequently used as a means for speculation. Basic reason: the option is more sensitive to price changes than the underlying asset itself. Consider for example a European call with strike $K = 50$, at a time t so near maturity that the value of the option is essentially $(S_t - K)_+$. Let $S_t = 60$ now, and consider what happens when S_t increases by 10% to 66. The value of the option increases from about $60 - 50 = 10$ to about $66 - 50 = 16$, an increase of 60%. Similarly if S_t decreases by 10% to 54 the value of the option decreases from 10 to 4, a loss of 60%. This calculation isn't special to a call:

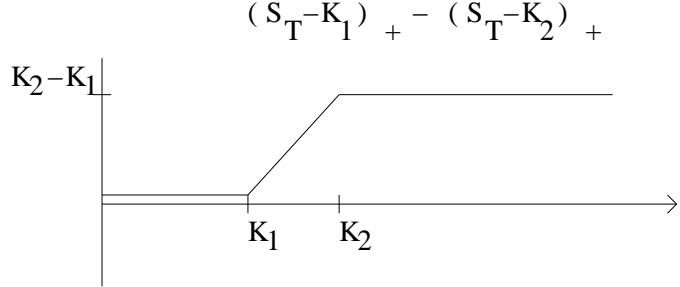


Figure 2: Payoff of a bull spread.

almost the same calculation applies to stock bought with borrowed funds. Of course there's a difference: the call has more limited downside exposure.

We assumed the time t was very near maturity so we could use the payoff $(S_T - K)_+$ as a formula for the value of the option. But the idea of the preceding paragraph applies even to options that mature well in the future. We'll study in this course how the Black-Scholes analysis assigns a value $c = c[S_t; T - t, K]$ to the option, as a function of its strike K , its time-to-maturity $T - t$ and the current stock price S_t . The graph of c as a function of S_t is roughly a smoothed-out version of the payoff $(S_t - K)_+$.

Don't be confused: our assertion that "the option is more sensitive to price changes than the underlying asset itself" does *not* mean that $\partial c / \partial S$ is bigger than 1. This expression, which gives the sensitivity of the option to change in the underlying, is called Δ . At maturity the call has value $(S_T - K)_+$ so $\Delta = 1$ for $S_T > K$ and $\Delta = 0$ for $S_T < K$. Prior to maturity the Black-Scholes theory will tell us that Δ varies smoothly from nearly 0 for $S_t \ll K$ to nearly 1 for $S_t \gg K$.

Forward contracts can also be used for speculation. Holding a portfolio of assets accomplishes two things: (i) it is a place to invest money you don't need now to meet future needs (e.g., saving for retirement) and (ii) you invest in assets you think will increase in value. But suppose you only want to accomplish the second objective and don't have need of the first. Forwards give you a means to take positions in assets you think will increase in value without tying this to the investment of cash. Equally important, they give you a means for taking a position in assets you think will go down in value or taking positions that reflect views on relative performance (long one set of assets and short another set; you may not have an opinion as to whether either will increase or decrease in value but you do believe the first set will outperform the second).

Arbitrage pricing. This class is about the pricing of derivative securities. We cannot of course predict the price of any single security. But when securities are related to one other (a call and a put on the same underlying, for example) their prices must also be related. The arguments we'll use to prove this involve the *absence of arbitrage*.

Why must arbitrages be absent? There are individuals and firms whose main activity is to identify arbitrages and take advantage of them. These investors use arguments like ours to guide their actions. (But: since true arbitrages are rare, they often also seek *statistical arbitrages*, i.e. opportunities that will most likely produce a gain, though there is some chance of a loss.) The actions of these investors influence prices, bringing them back into line and making it approximately true that the market has no arbitrages. Therefore the rest of us can price instruments using the absence of arbitrage.

Today we'll focus on the pricing of forwards, and put-call parity. The arguments we'll use are *model-free*: they do not require a model for the evolution of the underlying asset. Since we aren't assuming much, the arguments are relatively simple. Moreover their conclusions are very robust.

Soon – and for a good part of the semester – we'll turn to the pricing of options. In that setting there are few useful model-free results. We will, however, still make use of arbitrage-based arguments, by (i) assuming a probabilistic model for the evolution of the underlying, then (ii) applying the absence of arbitrage. This will lead us to the famous Black-Scholes option pricing formula, and far beyond.

Some pricing principles:

- If two portfolios have the same payoff then their present values must be the same.
- If portfolio 1's payoff is always at least as good as portfolio 2's, then present value of portfolio 1 \geq present value of portfolio 2.

We'll see presently that these principles must hold, because if they didn't the market would support arbitrage.

First example: value of a forward contract. We assume for simplicity:

- underlying asset pays no dividend and has no carrying cost (e.g. a non-dividend-paying stock);
- time value of money is computed using compound interest rate r , i.e. a guaranteed income of D dollars time T in the future is worth $e^{-rT}D$ dollars now.

The latter hypothesis amounts to introducing one more investment option:

Bond worth D dollars at maturity T

$$\begin{aligned} \text{buy a bond} &\leftrightarrow \text{hold a long bond} \\ &\leftrightarrow \text{lend } e^{-rT}D \text{ dollars, to be repaid at time } T \text{ with interest.} \end{aligned}$$

Consider these two portfolios:

Portfolio 1 – one long forward with maturity T and delivery price K , payoff $(S_T - K)$.

Portfolio 2 – long one unit of stock (present value S_0 , value at maturity S_T) and short one bond (present value $-Ke^{-rT}$, value at maturity $-K$).

They have the same payoff, so they must have the same present value. Conclusion:

$$\text{Present value of forward} = S_0 - Ke^{-rT}.$$

In practice, forward contracts are normally written so that their present value is 0. This fixes the delivery price, known as the *forward price*:

$$\text{forward price} = S_0 e^{rT} \text{ where } S_0 \text{ is the spot price.}$$

We can see why the “pricing principles” enunciated above must hold. If the market price of a forward were different from the value just computed then there would be an arbitrage opportunity:

$$\begin{aligned} \text{forward is overpriced} &\rightarrow \text{sell portfolio 1, buy portfolio 2} \\ &\rightarrow \text{instant profit at no risk} \\ \text{forward is underpriced} &\rightarrow \text{buy portfolio 1, sell portfolio 2} \\ &\rightarrow \text{instant profit at no risk.} \end{aligned}$$

In either case, market forces (oversupply of sellers or buyers) will lead to price adjustment, restoring the price of a forward to (approximately) its no-arbitrage value.

Two pieces of financial markets terminology that you may find confusing:

- Individuals commonly only borrow and lend one type of financial instrument – currency (e.g. dollars). But financial institutions borrow and lend all types of financial instruments. So in the above example, when we talked about “selling portfolio 2,” this involves borrowing the stock at time 0 and selling it, then buying the stock at time T in order to repay the borrowing. This is the same mechanism that is used when “short sellers” want to put on a position that will benefit from a decline in stock prices.
- Holding a bond is used as a synonym for lending money. Shorting a bond is used as a synonym for borrowing money. So in the above example, when we talk about buying portfolio 2, this means buying the stock and borrowing the money to buy the stock, and borrowing money is equivalent to shorting the bond (to see this consider that borrowing involves receiving Ke^{-rT} dollars now and paying K dollars at time T ; if you borrow the bond now you can sell it for Ke^{-rT} dollars and at time T repay the bond and receive K dollars, the bond’s principal). When we talk about selling portfolio 2, this means borrowing the stock to sell it and lending the money received from the sale until time T , and lending money is equivalent to buying the bond (to see this, consider that lending involves paying Ke^{-rT} dollars now and receiving K dollars at time T ; if you buy the bond now for Ke^{-rT} dollars you can redeem it at time T for the principal amount of K dollars).

Second example: put–call parity. Define

$$\begin{aligned} p[S_0, T, K] &= \text{price of European put when spot price is} \\ &\quad S_0, \text{strike price is } K, \text{maturity is } T \\ c[S_0, T, K] &= \text{price of European call when spot price is} \\ &\quad S_0, \text{strike price is } K, \text{maturity is } T. \end{aligned}$$

The Black-Scholes model gives formulas for p and c based on a certain model of how the underlying security behaves. But we can see now that p and c are related, without knowing anything about how the underlying security behaves (except that it pays no dividends and has no carrying cost). “Put-call parity” is the relation

$$c[S_0, T, K] - p[S_0, T, K] = S_0 - Ke^{-rT}.$$

To see this, compare

Portfolio 1 – one long call and one short put, both with maturity T and strike K ; the payoff is $(S_T - K)_+ - (K - S_T)_+ = S_T - K$.

Portfolio 2 – a forward contract with delivery price K and maturity T . Its payoff is also $S_T - K$.

These portfolios have the same payoff, so they must have the same present value. This justifies the formula.

Third example: The prices of European puts and calls satisfy

$$c[S_0, T, K] \geq (S_0 - Ke^{-rT})_+ \quad \text{and} \quad p[S_0, T, K] \geq (Ke^{-rT} - S_0)_+.$$

To see the first relation, observe first that $c[S_0, T, K] \geq 0$ by optionality – holding a long call is never worse than holding nothing. Observe next that $c[S_0, T, K] \geq S_0 - Ke^{-rT}$, since holding a long call is never worse than holding the corresponding forward contract. Thus $c[S_0, T, K] \geq \max\{0, S_0 - Ke^{-rT}\}$, which is the desired conclusion. The argument for the second relation is similar.

Financial context. Note the following hypotheses underlying our discussion:

- no transaction costs; no bid-ask spread;
- no tax considerations;
- unlimited possibility of long and short positions – no restriction on borrowing;
- same cost for borrowing and lending money;
- no charge for borrowing securities.

These are of course merely approximations to the truth (like any mathematical model). More accurate for large institutions than for individuals.

Note also some features of our discussion: We are simply reaping consequences of the hypothesis of no arbitrage. Conclusions reached this way don’t depend at all on what you think the market will do in the future. Arbitrage methods restrict the prices of (related)

instruments. On the other hand they don't tell an individual investor how best to invest his money. That's the issue of portfolio optimization, which requires an entirely different type of analysis and is discussed in the course Risk & Portfolio Management with Econometrics.

Importance of the forward price. Forwards have been designed to make it easy for investors with a view on price movements (or an existing position whose price risk they wish to hedge) to express that view without investing cash. When the investor changes his view or just wishes to realize his profits, it should be easy to reverse his position. In any case, an investor should be able to settle his position without having to actually take or make delivery under the forward contract, since the investor may just have a price view and not be a dealer in the actual underlying instrument. Forwards (and futures) are well-structured to accommodate these objectives, since an offsetting trade that takes place prior to the settlement date cancels the need for settlement.

This highlights the importance of the forward price, the delivery price that results in no upfront payment. If a forward trade was entered into on day 0 at F_0 =the forward price on day 0, and is offset on day t at F_t =the forward price on day t , then the resulting gain is precisely $F_t - F_0$, to be paid on the settlement date. Since these forward prices have been set so that no upfront payments are due, payments are made *only* on the settlement date.

More realistic assumptions about borrowing stock. We have implicitly been assuming up till now that the stock can be borrowed without paying a borrowing fee (since we assumed that we could currently sell the stock at price S_0 and buy it at time T at S_T and have not talked about any cost for borrowing the stock during that period). How realistic is that assumption? It's not totally unrealistic, because we have been assuming a non-dividend-paying stock, so we don't need to pay the stock lender for missing out on dividends. We don't have to pay the stock lender for any credit risk that we don't pay her back, because we can use the cash we receive for selling the stock as collateral to assure the lender she will receive the stock back. The holder can't expect to be compensated for having her money tied up in the stock – the expected increase in stock price is supposed to be her return for that. But even with these considerations, it is common for holders of stock to require some fee for lending their stock (if for no other reason than that holders of a stock, who want to see the price go up, need some compensation for aiding short sellers who by selling stock are helping to drive the price of the stock down). The fee may be quite small – a rate of $\frac{1}{2}\%$ per year is not uncommon – but can also be considerably larger, e.g. if there is a big demand for borrowing the stock (if many people desire to sell the stock short, there may be a shortage of stock available for borrowing).

What is the impact of this fee? It lowers the cost of buying portfolio 2 in our discussion of forwards. The reason is that while holding the stock, you can lend it out and earn the stock borrowing fee. Similarly, the fee raises the cost of selling portfolio 2, since when you sell stock you must also pay the borrowing fee. If we assume a continuously compounded stock borrowing rate of q per annum, then the value at time 0 of lending the stock from time 0 to time T is (by definition) $S_0(1 - e^{-qT})$. The cost at time 0 of going long portfolio 2 is therefore $(S_0 - Ke^{-rT}) - S_0(1 - e^{-qT}) = S_0e^{-qT} - Ke^{-rT}$. So the impact is to make the present value of a forward with delivery price K be $S_0e^{-qT} - Ke^{-rT}$. The forward price

(the choice of K that makes this expression zero is now $S_0 e^{(r-q)T}$. (Compare with Hull's discussion in section 5.6 on forwards on assets with known yield.)

How do we handle a dividend paying stock? Dividends are paid at fixed dates (usually once a quarter) and are not known in advance. For simplicity, we will approximate this effect by assuming that the market can project dividends with good accuracy (a reasonable assumption over short time periods), while noting that any uncertainty will widen the bounds within which arbitrage can determine the forward price. We will approximate the dividend by an annualized, continuously compounded rate q .

Instead of constructing portfolio 2 by buying one unit of the stock, we now buy e^{-qT} units of the stock and continuously reinvest all dividends in the stock. By the end of period T , this will result in our holding exactly the one unit of the stock which we need to deliver into the forward contract of portfolio 1. If we are selling portfolio 2, we borrow e^{-qT} units of the stock and continuously borrow more units at the dividend rate q . (A fully detailed version of this can be found in Baxter & Rennie, p. 107. Also see Hull, section 5.6).

When we have both borrowing costs of the stock and dividend, we just use a q which represents the dividend rate plus the borrowing cost. (Institutional detail – the contract between the stock borrower and lender can either call for the borrower to return to the lender (1) just the stock or (2) the stock plus all dividends paid during the borrowing period. In the first case, the borrowing rate paid will equal the expected dividend rate plus a borrowing add-on, to compensate the lender for missing dividends. In the second case, the borrowing rate should be just the same as in our non-dividend-paying stock case, but the actual payment by the borrower will include the dividend.)

Forwards on an asset with continuous yield. The preceding discussion applies to *any* asset with a continuous return. If q is the rate of return (annualized and continuously compounded), the present value of a forward at a settlement price of K given on an asset currently priced at S_0 is $S_0 e^{-qT} - K e^{-rT}$ and the forward price (i.e. the settlement price for which the present value equals zero) is $S_0 e^{(r-q)T}$. Our original example of a non-dividend-paying stock with a zero borrowing cost corresponds to the special case $q = 0$. Here are some other important examples:

Forwards on stock indices. The stock index forward can be arbitAGED by a portfolio of all the individual stocks in the index with asset weights equal to those of the index (Hull 5.9). Dividends and borrowing costs of this basket can be estimated as weighted averages of the dividends and borrowing costs of each individual stock.

Forwards on foreign exchange rates. Say you have a forward agreement to exchange X units of one currency for KX units of another currency (example $X = 1\text{MM}$, $K = 1.25$, you have a forward agreement to exchange 1MM Euros for 1.25MM Dollars). Buying Portfolio 2 for the arbitrage consists of (1) buying $1\text{MM} e^{-qT}$ Euros now, investing them at the Euro lending rate of q and having 1MM Euros at the end of time T , and (2) borrowing $1.25\text{MM} e^{-rT}$ Dollars now, borrowed at the Dollar lending rate of r , and having 1.25MM Dollars at the end of time T . Since this is equivalent at the end of T to the forward, the present value of the forward must equal the current cost of Portfolio 2, which is $1\text{MM} e^{-qT} S_0 - 1.25\text{MM} e^{-rT}$, where S_0 is the current exchange rate for Euros into Dollars. This

is clearly just an example of our general formula. Compare with Hull 5.10. The forward exchange rate that makes this present value equal to 0 is $S_0 e^{(r-q)T}$.

Forward contract to exchange one asset for another. There's nothing special about one of the assets in this last example being dollars. We could consider a forward contract to exchange Yen for Euros and its present value, by the exact same reasoning as above, would be $e^{-qT}S_0 - e^{-rT}K$, where S_0 is the current exchange rate between Yen and Euros, K the settlement exchange rate, q the borrowing rate for Euros, and r the borrowing rate for Yen. This formula gives the present value in Yen; to get a Dollar present value you need to multiply by the current Dollar/Yen exchange rate. We could equally well value an exchange contract between any two assets: gold and diamonds, oil and cattle, oil and Yen, etc. Compare with Hull 25.13 (that section is about options to exchange one asset for another, but the reasoning offered there applies also to forwards).

Forwards on commodities. Traditional treatments, such as Hull 5.11, distinguish between commodities that are primarily investment assets, such as gold and silver, and commodities that are primarily utilized for consumption, such as oil. This distinction is parallel to the difference between non-dividend-paying stocks and currencies. An investment asset (like a non-dividend-paying stock) can be expected to have a very low borrowing rate q , because the only demand to borrow the asset comes from those wishing to sell it short (it might be even lower for an asset like gold that has associated storage costs which the borrower is taking over from the investor). A consumption asset, just like a dividend-paying stock or a currency, can be expected to have a relatively high borrowing cost since there is diversified demand to borrow it. But (unlike currency) there is usually no direct way to see the borrowing cost of a commodity – it needs to be backed out of the forward price.

Usually investment assets will have a borrowing rate q that's lower than the risk-free interest rate (on currency), r . Therefore the forward price $S_0 e^{(r-q)T}$ of an investment asset is usually higher than the current price. For commodities, this situation is known as *contango*. A consumption asset might well have a borrowing cost q that's higher than the risk-free interest rate r . If this is the case then its forward price $S_0 e^{(r-q)T}$ will be lower than the current price. For commodities, this situation is known as *backwardation*.

Since there is usually no direct way to borrow a commodity, there won't be arbitrage opportunities between the borrowing cost implied by the forward price and the borrowing cost coming from another market. But the forwards market can be used as a means for dealers in commodities to borrow the commodity, so this is still a useful relationship to consider. For example, a dealer in oil may need to temporarily borrow oil to make a scheduled delivery ahead of the arrival of a scheduled shipment. He can do this by borrowing dollars, using the dollars to buy oil, and selling oil forward to the date on which his shipment is scheduled. On the scheduled shipment date, he uses the oil he receives to deliver against the forward and takes the cash he receives in exchange to pay off his dollar loan. He has taken no price risk to changes in oil prices, and he has locked in a borrowing cost for the oil through the combination of the dollar borrowing and the forward transaction.

The forward price of oil determines an effective borrowing rate for oil. To see how, let S_0 be the current cash price of one barrel of oil, F_0 the current forward price for delivery at time T , and r the risk-free interest rate. Consider the following trade: at time 0 you

- borrow S_0 dollars and use it to buy a barrel of oil; simultaneously you enter into a forward contract at time 0 to sell X barrels of oil at time T at the forward rate. The correct choice of X will become clear in a moment; it is $X = S_0 e^{rT} / F_0$.

At time T , you:

- fulfill the forward contract, delivering $S_0 e^{rT} / F_0$ barrels of oil and receiving $S_0 e^{rT}$ dollars; then use these dollars to repay the loan (including the interest).

Since the forward contracts were at the forward rate, their cost is zero. So this trade has the effect of *borrowing a barrel of oil at time 0 and repaying the loan by delivering e^{dT} barrels of oil at time T* , where

$$dT = \ln \left(S_0 e^{rT} / F_0 \right).$$

This d is the effective borrowing rate for oil. Rewriting its definition in the form $F_0 = S_0 e^{(r-d)T}$, we recognize that d plays the same role as the borrowing rate q discussed earlier.

The same argument can be used to create a synthetic borrowing rate for a currency or a stock. But in these cases, a direct borrowing market also exists and arbitrage will drive the synthetic rate and direct rate towards equality.

A word about interest rates. In the real world interest rates change unpredictably. And the rate depends on maturity. In discussing forwards and European options this isn't particularly important: all that matters is the cost "now" of a bond worth one dollar at maturity T . Up to now we wrote this as e^{-rT} . When multiple borrowing times and maturities are being considered, however, it's clearer to use the notation

$$B(t, T) = \text{cost at time } t \text{ of a risk-free bond worth 1 dollar at time } T.$$

In a constant interest rate setting $B(t, T) = e^{-r(T-t)}$. If the interest rate is non-constant but deterministic – i.e. known in advance – then an arbitrage argument shows that $B(t_1, t_2)B(t_2, t_3) = B(t_1, t_3)$. If however interest rates are stochastic – i.e. if $B(t_2, t_3)$ is not known at time t_1 – then this relation must fail, since $B(t_1, t_2)$ and $B(t_1, t_3)$ are (by definition) known at time t_1 .

Since our results on forwards, put-call parity, etc. used only one-period borrowing, they remain valid when the interest rate is nonconstant and even stochastic. For example, the value at time 0 of a forward contract with delivery price K is $S_0 - KB(0, T)$ where S_0 is the spot price. We could also use the notation r_T for the constant interest rate that gives the right result on the time interval $(0, T)$; it is defined by $B(0, T) = e^{-r_T T}$. Note however that if the interest rate is not constant then $r_{T_1} \neq r_{T_2}$ for $T_1 \neq T_2$.

Forwards versus futures. A future is a lot like a forward contract – its writer must sell the underlying asset to its holder at a specified maturity date. However there are some important differences:

- Futures are standardized and traded, whereas forwards are not. Thus a futures contract (with specified underlying asset and maturity) has a well-defined “future price” that is set by the marketplace. At maturity the future price is necessarily the same as the spot price.
- Futures are “marked to market,” whereas in a forward contract no money changes hands till maturity. Thus the value of a future contract, like that of a forward contract, varies with changes in the market value of the underlying. However with a future the holder and writer settle up daily while with a forward the holder and writer don’t settle up till maturity.

The essential difference between futures and forwards involves the timing of payments between holder and writer: daily (for futures) versus lump sum at maturity (for forwards). Therefore the difference between forwards and futures has a lot to do with the time value of money. If interest rates are constant – or even nonconstant but deterministic – then an arbitrage-based argument shows that the forward and future prices must be equal. Here’s a quick sketch of the argument (see also Technical Note 24 on Hull’s website www.rotman.utoronto.ca/~hull/TechnicalNotes for a different organization of what is fundamentally the same argument).

The forward price and futures price must be the same on the settlement date, and also one day prior to the settlement date, since there is no difference in these cases as to when payments are received. This gives us the first step of an argument by mathematical induction. Now we must prove the *inductive step*, by showing that if the forward price equals the futures price on day N then they are also equal on day $N - 1$. Let’s call P the common price on day N ; let F be the forward price on day $N - 1$ and let G be the futures price on day $N - 1$. We are assuming that rates are deterministic, so we know now the interest rate for investing cash on day N with maturity equal to the settlement date T ; let’s call that rate R . We’ll write $\tau = T - N$ for the time from day N till settlement. Now consider the following two investment strategies:

Strategy 1 – buy $e^{-R\tau}$ units of the future on day $N - 1$ (by definition, no money changes hands), then liquidate the position on day N and invest the gain $e^{-R\tau}(P - G)$ at rate R until the settlement date. This results in a gain of $e^{R\tau}e^{-R\tau}(P - G) = P - G$ on the settlement date.

Strategy 2 – buy a forward at the forward price on day $N - 1$ (again, no money changes hands) and sell it on day N . This this produces a gain of $P - F$, received on the settlement date.

Each strategy requires no initial investment. Their outcome is not deterministic¹ (since we don’t know P on day $N - 1$). But the outcome is the *same*. We conclude that $F = G$ as

¹The previous version of these notes was incorrect at this point.

follows. If $F > G$, then combining Strategy 1 with the opposite of Strategy 2 (buying $e^{-R\tau}$ futures and selling a forward on day $N - 1$, etc) gives an arbitrage, since it costs nothing initially and it produces a risk-free gain of $F - G$ on the settlement date. If $F < G$, then combining Strategy 2 with the opposite of Strategy 1 gives an arbitrage by essentially the same argument. So absence of arbitrage implies $F = G$.

If interest rates are stochastic, the arbitrage-based relation between forwards and futures breaks down, and forward prices can be different from future prices. In practice they are different, but usually not much so. Later in the semester, when we discuss interest rate derivatives, we'll return to the difference between futures and forwards. In particular we'll give a different proof of the equality of forward and futures prices when interest rates are deterministic, and we'll look at how the relationship changes when interest rates are stochastic. Until we reach that part of the course, however, we'll ignore the difference and treat forwards and futures as synonymous.

To better understand the reason for the structure of forwards and futures, we need to appreciate how these markets manage credit risk. As we've seen, forwards and futures eliminate the need to invest cash when taking a position on price movements. However this introduces an important element of credit risk, since cash investment also serves the purpose of assuring that the investor cannot walk away from losses due to adverse price moves. Futures markets deal with the credit risk by utilizing a trusted institution, the futures exchange, backed by the credit of all its members, to act as a guarantor of credit to all investors who transact in futures dealt on the exchange. It works as follows.

When a buyer and seller agree on a futures price, that ends their relationship with one another; the futures exchange immediately inserts itself as the counterparty to both buyer and seller. This entails no price risk to the futures exchange, since it has exactly offsetting transactions with the buyer and seller. It gives both the buyer and seller complete assurance that their contracts will be honored, since the futures exchange has such well-established credit. Furthermore, it makes for complete flexibility to the buyer and seller when and if they want to unwind their positions, since they don't have to reach an agreement with their original counterparty. They need simply find any other seller or buyer for an offsetting transaction – no matter who they transact with, their real counterparty is the futures exchange, which will net all its transactions with the individual investor.

The futures exchange does need to bear the credit risk that may result from buyers or sellers not being able to meet their obligations when prices move. The futures exchange deals with this by requiring collateral to cover possible price moves and by requiring cash payments every time the price moves. This is the reason that cash must change hands immediately as the futures prices moves. If an investor does not pay cash when required by a price move (this is called failing to meet a margin call), the futures exchange will take over the position and close it out, using the investor's collateral to cover any ensuing loss. The need for liquid contracts that can easily be closed out in the event of a failure to meet margin call is one reason that futures exchanges deal only in standardized contracts with a limited range of specifications.

By contrast, forward contracts can be more individually tailored and do not require constant cash flows. Investors are generally only willing to enter into such contracts with

counterparties of very high credit worthiness, as this is the only assurance they have that their contracts will be honored. As a result, market makers in forward contracts are limited to very large, well-capitalized institutions. These market makers also need to be expert in assessing and managing credit risk, since they bear the risk of investors not meeting their obligations under the forward contracts. We'll discuss some of the measures they take when we talk about credit risk at the end of the semester.

A word about taxes. Tax considerations are not always negligible. Here is an example, closely related to put-call parity.

Constructive sales. An investor holds stock in XYZ Corp. His stock has appreciated a lot, and he thinks it's time to sell, but he wishes to postpone his gain till next year when he expects to have losses to offset them. Prior to 1997 he could have (1) kept his stock, (2) bought a put (one year maturity, strike K), (3) sold a call (one year maturity, strike K), and (4) borrowed Ke^{-rT} . The value of this portfolio at maturity is $S_T + (K - S_T)_+ - (S_T - K)_+ - K = 0$. Since his position at time T is valueless and risk-free, he would have effectively "sold" his stock. Since the present value of items (1)-(4) together is 0, the combined value of the long put, short call, and loan must be the present value of the stock. Thus the investor would have effectively sold the stock for its present market value, while postponing realization of the capital gain till the options matured.

The tax law was changed in 1997 to treat such a transaction as a "constructive sale," eliminating its attractiveness (the capital gain is no longer postponed). A related strategy is still available however: by combining puts and calls with different maturities, an investor can take a position that still has some risk (thus avoiding the constructive sale rule) while locking in most of the gain and avoiding any capital gains tax till the options mature.