Derivative Securities – Fall 2007 – Section 8
Notes by Robert V. Kohn, extended and improved by Steve Allen.
Courant Institute of Mathematical Sciences.

American and exotic options. We have thus far focused on European options. This week's topic is the valuation and hedging of American and exotic options.

This short document (4 pages) discusses only American options. *Please also read Steve Allen's Section 8 notes* (posted on Blackboard); they focus mainly on (a) the numerical valuation of path-dependent options, and (b) creation of a binomial tree that's consistent with an observed volatility skew/smile.

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American options. American options are different in that they permit early exercise: the holder of an American option can exercise it at any time up to the maturity T. Of the options actually traded in the market, the majority are American rather than European.

Clearly an American option is at least as valuable as the analogous European option, since the holder has the option to keep it to maturity.

Fact: An American call written on a stock that earns no dividend has the same value as a European call; early exercise is never optimal. To see why, suppose the strike price is K and consider the value of the American option "now," at some time $t < T$. Exercising the option now achieves a value at time t of $s_t - K$. Holding the option to maturity achieves a value at time t equal to that of a European call, $c[s_t, K, T - t]$. Without using the Black-Scholes formula (thus without assuming lognormal stock dynamics) we know the value of a European call is at least that of a forward with the same strike and maturity. Thus holding the option to maturity achieves a value at time t of at least $s_t - e^{-r(T-t)}K$. If $r > 0$ this is larger than $s_t - K$. So early exercise is suboptimal, as asserted.

The preceding is in some sense a fluke. When the underlying asset pays a dividend early exercise of a call can be optimal. But the simplest example where early exercise occurs is that of a put on a non-dividend-paying stock:

Fact: An American put written on a stock that earns no dividend can have a value greater than that of the associated European put; early exercise can be optimal. To see why, consider once again the value of the American option "now," at some time $t < T$. Exercising the option now achieves value $K - s_t$. Holding it to maturity achieves a value at time t equal to that of a European put, $p[s_t, K, T - t]$. Assuming lognormal stock price dynamics, p is given by the Black-Scholes formula, and its graph as a function of spot price s_t is shown in the figure.

The important point is that $p[s_t, K, T - t]$ is strictly less than $K - s_t$ when $s_t \ll K$. This is immediate from the Black-Scholes formula, since $p = Ke^{-r(T-t)}N(-d_2) - s_tN(-d_1) \approx Ke^{-r(T-t)} - s_t$ when $s_t \ll K$, since $d_1 \to -\infty$ and $d_2 \to -\infty$ as $s_t/K \to 0$. Briefly: if

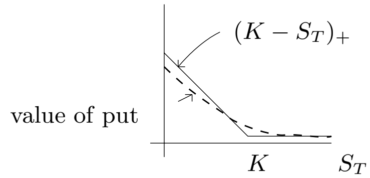

Figure 1: The value of a European put lies below the payoff when $s \ll K$.

$s_t \ll K$ then the put is deep in the money, and the (risk-neutral) probability of it being out of the money at time T is vanishingly small; therefore the value of the put is almost the same as the value of a short forward. In such a situation we are better off exercising the option at time t than holding it to maturity. (This does not show that exercise at time t is optimal, but it does show holding the option to maturity is not optimal.)

For European options we have three different (but related) valuation techniques: (a) working backward through the binomial tree; (b) evaluating the discounted expected payoff (using the risk-neutral version of the price process); and (c) solving the Black-Scholes PDE. Each of the three viewpoints can be extended to American options. We assume for simplicity that the underlying asset pays no dividends.

Valuation using a binomial tree. This is perhaps the simplest approach, conceptually and numerically. We can use the same recombinant binomial tree as for a European option. (Remember: pricing is done using the risk-neutral process. If the underlying asset is lognormal with volatility σ then a convenient choice of the parameters defining the tree is $u = \exp[(r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}]$, $d = \exp[(r - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}]$, where r is the risk-free rate). But since early exercise is permitted, we must ask at each node: is the option worth more "alive" or "dead"? If the option is worth more dead, then it should be exercised (by its holder) whenever the price arrives at that node. For example, consider the pricing of an American option with payoff $f(s)$ using a two-period recombinant tree:

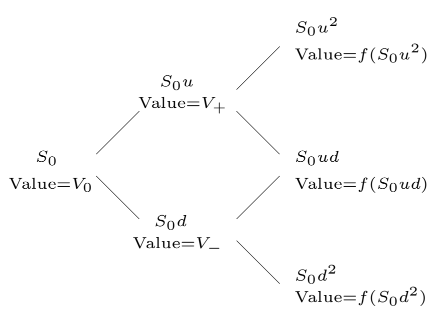

Figure 2: Valuation of an American option using a binomial tree.

When the stock price is s_0u the option is worth

$$f(s_0u) \text{ dead, and } e^{-r\delta t}[qf(s_0u^2) + (1-q)f(s_0ud)] \text{ alive.}$$

Allowing for both possibilities the value of the option at s_0u is

$$V_+ = \max\{f(s_0u), e^{-r\delta t}[qf(s_0u^2) + (1-q)f(s_0ud)]\}.$$

Similarly, when the stock price is s_0d the value is

$$V_- = \max\{f(s_0d), e^{-r\delta t}[qf(s_0ud) + (1-q)f(s_0d^2)]\}.$$

The value at the initial time is obtained by repeating the process:

$$V_0 = \max\{f(s_0), e^{-r\delta t}[qV_+ + (1-q)V_-]\}.$$

Our example has only two time periods, but a binomial tree of any size is handled similarly.

Valuation using the discounted expected payoff. For a European option, we saw that the value assigned by the binomial tree was expressible in the form $e^{-rT}E_{\rm RN}[f(s(T))]$. A similar calculation applies to the American option – however $f(s(T))$ must be replaced by the value realized *at exercise*: the value of the option is $E_{\rm RN}\left[e^{-r\tau}f(s(\tau))\right]$ where τ is the exercise time. Once we've worked backward through the tree we know how to determine τ – for each realization of the risk-neutral process, it's the first time that realization reaches a node of the tree associated with early exercise (or T, if that realization does not reach an "early-exercise" node).

Actually, this viewpoint can also be used, at least conceptually, to *determine* the early-exercise criterion, without working backward through the tree. In fact,

$$\text{Value} = \max_{\text{exercise rules}} E_{\rm RN}\left[e^{-r\tau}f(s(\tau))\right].$$

In other words the exercise rule selected by backsolving the binomial tree is the one that maximizes the discounted expected payoff. An honest proof of this fact is not trivial – mainly because it requires formalization of what one means by an "exercise rule." But heuristically: any exercise rule determines a hedging strategy, i.e. a synthetic option that is available in the marketplace. So the max over exercise rules gives a lower bound for the value of the option. Our strategy of working backward through the tree gives an upper bound. The two bounds agree since the value obtained by working backward through the tree is associated with a special exercise rule.

Valuation using a PDE. (The following material is not in Hull; you can find a brief summary in the "student guide" by Wilmott-Howison-Dewynne; it will not be on the HW or exam.) For a European option the continuous-time limit of working backward through the tree amounts to solving the Black-Scholes PDE for $t < T$, with final data $f(s)$ at $t = T$. There is an analogous statement for an American option, however the PDE is replaced by a *free boundary problem*:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V}{\partial s} - rV \le 0,$$

$$V(s,t) \ge f(s),$$

and

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V}{\partial s} - rV = 0 \quad \text{or} \quad V(s,t) = f(s).$$

The logic behind the first inequality is this: in our derivation of the Black-Scholes PDE, the crucial juncture was when we saw that the choice $\phi = \partial V/\partial s$ made $d(V - \phi s)$ deterministic:

$$d(V - \phi s) = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2\right) dt.$$

We concluded, by the principle of no arbitrage, that this must equal $r(V - \phi s)dt$. But that arbitrage argument assumed that you continued to hold the option. In the present context, where early exercise is permitted, the absence of arbitrage gives a weaker conclusion: the deterministic portfolio $(V - \phi s)$ can grow *no faster* than the risk-free rate. Thus

$$\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2\right) \leq r\left(V - \frac{\partial V}{\partial s}s\right);$$

this is our first inequality. The logic behind the second inequality is obvious: the value is no smaller than can be realized by immediate exercise. The third relation simply says that one of the first two relations always holds – because for any given (s,t) the optimal strategy involves either holding the option a little longer (in which case the Black-Scholes equation applies) or exercising it immediately.

We call this a free-boundary problem because the value is still governed by the Black-Scholes PDE in *some* region of the (s,t) plane – the region where immediate exercise isn't optimal – however this region isn't given as data but must be found as part of the problem. Schematically:

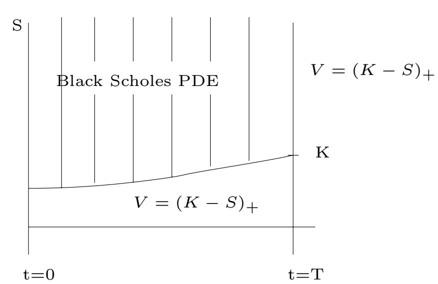

Figure 3: Schematic of the free boundary problem whose solution values an American put.

One can show that V and $\Delta = \partial V/\partial s$ are both continuous across the free boundary. Of course, on the "exercise" side of the boundary $V = f(s)$ and $\partial V/\partial s = f'(s)$ are known, giving two boundary conditions. If the domain of the PDE were known then just one boundary condition would be permitted; but the domain isn't known, and the extra boundary condition serves to fix the free boundary.