

## Derivative Securities – Fall 2007– Section 3

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**Multiperiod Binomial Trees.** We turn to the valuation of derivative securities in a time-dependent setting. We focus for now on multi-period binomial models, i.e. binomial trees. This setting is simple enough to let us do everything explicitly, yet rich enough to approximate many realistic problems.

The material covered in this section is very standard (and very important). The treatment here tracks closely with Baxter and Rennie (Chapter 2). Hull addresses this topic in Chapter 11 (6th edition). Another good treatment is that of Jarrow and Turnbull (Chapter 5, 2nd edition), which includes many examples. In the Section 4 notes we'll discuss how the parameters should be chosen to mimic the conventional (Black-Scholes) hypothesis of lognormal stock prices, and we'll pass to the continuous-time limit.

Binomial trees are widely used in practice, in part because they are easy to implement numerically. (Also because the scheme can easily be adjusted to price American options.) For a nice discussion of alternative numerical implementations, see the article “Nine ways to implement the binomial method for option valuation in Matlab,” by D.J. Higham, SIAM Review 44 (2002) 661-677.

As we saw in Section 2, an option can be replicated by trading the underlying, or alternatively by trading forwards or futures on the underlying. We'll focus first on replication by trading the underlying (since this is the first thing you'll see in most books). Then we'll make a second pass, replicating with futures (the viewpoint emphasized by Steve Allen's version of these notes).

The practical question is not how to replicate an option but rather how to hedge it. But replication and hedging are intimately connected. Simple example: when you sell a call with strike  $K$  and maturity  $T$  you receive cash for it now, but you owe  $(s_T - K)_+$  to the holder when the option matures. We'll explain (in the binomial setting) how the apparent uncertainty in your final-time obligation can be entirely eliminated (hedged) by pursuing a suitable trading strategy. It is, of course, the strategy that replicates your final-time obligation  $(s_T - K)_+$ .

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**First pass: trading the underlying.** A multi-period binomial model generalizes the single-period binomial model we considered in Section 2. It has

- just two securities: a risky asset (a “stock,” paying no dividend) and a riskless asset (“bond”);
- a series of times  $0, \delta t, 2\delta t, \dots, N\delta t = T$  at which trades can occur;
- interest rate  $r_i$  during time interval  $i$  for the bond;
- a binomial tree of possible states for the stock prices.

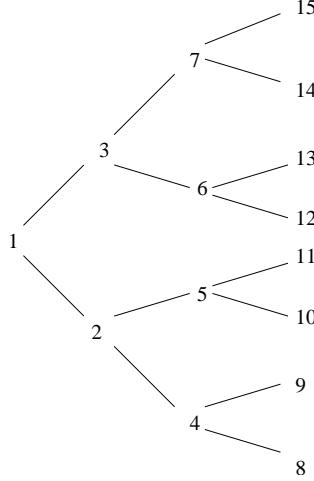


Figure 1: States of a non-recombining binomial tree.

The last statement means that for each stock price at time  $j\delta t$ , there are two possible values it can take at time  $(j + 1)\delta t$  (see Figure 1).

The interest rate environment is described by specifying the interest rates  $r_i$ . We restrict our attention for now to the case of a constant interest rate:  $r_i = r$  for all  $i$ .

The stock price dynamics is described by assigning a price  $s_j$  to each state in the tree. Strictly speaking we should also assign (subjective) probabilities  $p_j$  to the branches (the two branches emerging from a given node should have probabilities summing to 1): see Figure 2.

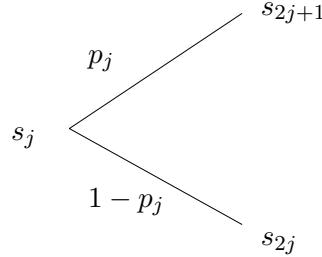


Figure 2: One branch of a binomial tree.

Actually, we will make no use of the subjective probabilities  $p_j$ ; our arguments are based on arbitrage, so they depend only on the list of possible states not on their probabilities. However our pricing formula will make use of *risk-neutral probabilities*  $q_j$ . These “look like” subjective probabilities, except that they are determined by the stock prices and the interest rate.

Our stock prices must be “reasonable” in the sense that the market support no arbitrage. Motivated by the one-period model, we (correctly) expect this condition to take the form:

- starting from any node, the stock price may do better than or worse than the risk-free rate during the next period.

In other words,  $s_{2j} < e^{r\delta t} s_j < s_{2j+1}$  for each  $j$ .

The tree in Figure 1 is the most general possible. At the  $n$ th time step it has  $2^n$  possible states. That's a lot of states, especially when  $n$  is large. It's often convenient to let selected states have the same prices in such a way that the list of distinct prices forms a *recombinant tree*. Figure 3 gives an example of a 4-stage recombinant tree, with stock prices marked for each state: (A recombinant tree has just  $n + 1$  possible states at time step  $n$ .)

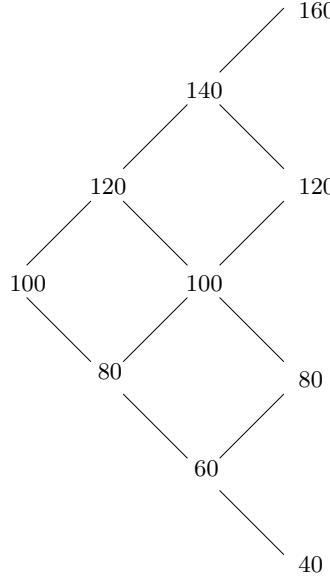


Figure 3: A simple recombinant binomial tree.

A special class of recombinant trees is obtained by assuming the stock price goes up or down by fixed multipliers  $u$  or  $d$  at each stage: see Figure 4. This last class may seem terribly special relative to the general binomial tree. But we shall see it is general enough for many practical purposes – just as a random walk (consisting of many steps, each of fixed magnitude but different in direction) can approximate Brownian motion. And it has the advantage of being easy to specify – one has only to give the values of  $u$  and  $d$ .

It may seem odd that we consider a market with just one stock, when real markets have many stocks. But our goal is to price contingent claims based on considerations of arbitrage. If we succeed using just these two instruments (the stock and the riskless bond) then our conclusions necessarily apply to any larger market containing both instruments.

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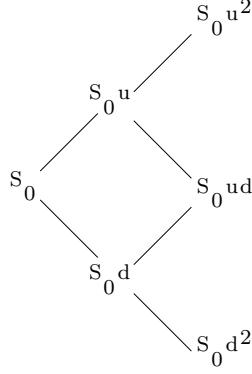


Figure 4: A multiplicative recombinant binomial tree.

Our goal is to determine the value (at time 0) of a contingent claim. We will consider American options later; for the moment we consider only European ones, i.e. early redemption is prohibited. The most basic examples are European calls and puts (payoffs:  $(S_T - K)_+$  and  $(K - S_T)_+$  respectively). However our method is much more general. What really matters is that *the payoff of the claim depends entirely on the state of stock process at time T*.

Let's review what we found in the one-period binomial model. Our multiperiod model consists of many one-period models, so it is convenient to introduce a flexible labeling scheme. Writing "now" for what used to be the initial state, and "up, down" for what used to be the two final states, our risk-neutral valuation formula was

$$f_{\text{now}} = e^{-r\delta t} [q f_{\text{up}} + (1 - q) f_{\text{down}}]$$

where

$$q = \frac{e^{r\delta t} s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}.$$

Here we're writing  $f_{\text{now}}$  for what we used to call  $V(f)$ , the (present) value of the contingent claim worth  $f_{\text{up}}$  or  $f_{\text{down}}$  at the next time step if the stock price goes up or down respectively. This formula was obtained by replicating the payoff with a combination of stock and bond; the replicating portfolio used

$$\phi = \frac{f_{\text{up}} - f_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}$$

units of stock, and a bond worth  $f_{\text{now}} - \phi s_{\text{now}}$ .

The valuation of a contingent claim in the multiperiod setting is an easy consequence of this formula. We need only "work backward through the tree," applying the formula again and again.

Consider, for example, the four-period recombinant tree shown in Figure 3. (This example, taken straight from Baxter and Rennie, has the nice feature of very simple arithmetic.) Suppose the interest rate is  $r = 0$ , for simplicity. Then  $q = 1/2$  at each node (we chose the

prices to keep this calculation simple). Let's find the value of a European call with strike price 100 and maturity  $T = 3\delta t$ . Working backward through the tree:

- The values at maturity are  $(S_T - 100)_+ = 60, 20, 0, 0$  respectively.
- The values one time step earlier are 40, 10, and 0 respectively, each value being obtained by an application of the one-period formula.
- The values one time step earlier are 25 and 5.
- The value at the initial time is 15.

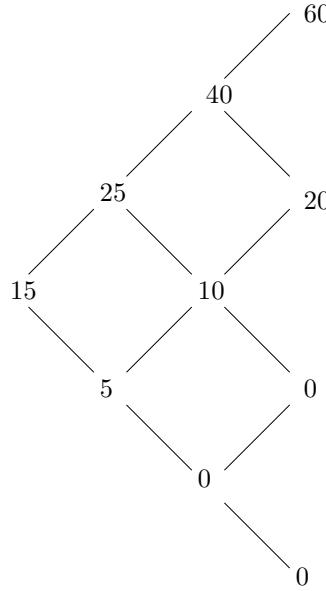


Figure 5: Value of the option as a function of stock price state.

Easy. But is it right? Yes, because these values can be replicated. However the replication strategies are more complicated than in the one-period case: the replicating portfolio must be adjusted at each trading time, taking into account the new stock price.

Let's show this in the example. Using our one-period rule, the replicating portfolio starts with  $\phi = (25 - 5)/(120 - 80) = .5$  units of stock, worth  $.5 \times 100 = 50$  dollars, and a bond worth  $15 - 50 = -35$ . (Viewed differently: the investment bank that sells the option collects 15 dollars; it should borrow another 35 dollars, and use these  $15+35=50$  dollars to buy 1/2 unit of stock.) The claim is that by trading intelligently at each time-step we can adjust this portfolio so it replicates the payoff of the option no matter what the stock price does. Here is an example of a possible history, and how we would handle it:

**Stock goes up to 120.** The new  $\phi$  is  $(40 - 10)/(140 - 100) = .75$ , so we need another .25 units of stock. We must buy this at the present price, 120 dollars per unit, and we do it by borrowing 30 dollars. Thus our debt becomes 65 dollars.

**Stock goes up again to 140.** The new  $\phi$  is  $(60 - 20)/(160 - 120) = 1$ , so we buy another .25 unit at 140 dollars per unit. This costs another 35 dollars, bringing our debt to 100 dollars.

**Stock goes down to 120.** At maturity we hold one share of stock and a debt of 100. So our portfolio is worth  $120 - 100 = 20$ , replicating the option. (Put differently: if the investment bank that sold the option followed our instructions, it could deliver the unit of stock, collect the 100, pay off its loan, and have neither a loss nor a gain.)

That wasn't a miracle. We'll explain it more formally pretty soon. But here's a second example – a different possible history – to convince you:

**Stock goes down to 80.** The revised  $\phi$  is  $(10 - 0)/(100 - 60) = .25$ . So we should sell  $1/4$  unit stock, receiving  $80/4=20$ . Our debt is reduced to 15.

**Stock goes up to 100.** The new  $\phi$  is  $(20 - 0)/(120 - 80) = .5$ . So we must buy  $1/4$  unit stock, spending  $100/4=25$ . Our debt goes up to 40.

**Stock goes down again to 80.** We hold 40 dollars worth of stock and we owe 40. Our position is worth  $40 - 40 = 0$ , replicating the option, which is worthless since it's out of the money. (Different viewpoint: the investment bank that sold the option can liquidate its position, selling the stock at market and using the proceeds to pay off the loan. This results in neither a loss nor a gain.)

Notice that the portfolio changes from one time to the next but the changes are *self-financing* – i.e. the total value of the portfolio before and after each trade are the same. (The investment bank neither receives or spends money except at the initial time, when it sells the option.)

Our example shows the importance of tracking  $\phi$ , the number of units of stock to be held as you leave a given node. It characterizes the replicating portfolio (the “hedge”). Its value is known as the *Delta* of the claim. Thus:

$$\Delta_{\text{now}} = \text{our } \phi = \frac{f_{\text{up}} - f_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}.$$

To understand the meaning  $\Delta$ , observe that as you leave node  $j$ ,

$$\text{value of claim at a node } j = \Delta_j s_j + b_j$$

by definition of the replicating portfolio (here  $b_j$  is the value of the bond holding at that moment). If the value of the stock changes by an amount  $ds$  while the bond holding stays fixed, then the value of the replicating portfolio changes by  $\Delta ds$ . Thus  $\Delta$  is a sort of derivative:

$$\Delta = \text{rate of change of replicating portfolio value, with respect to change of stock price.}$$

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**Second pass: trading futures.** An option on the underlying  $s$  with maturity  $T$  (for example a call, which pays  $(s_T - K)_+$  at time  $T$ ) can equally be viewed as an option on the forward price for delivery at time  $T$ , since the forward price agrees with the spot price at time  $T$ . But when we trade forwards or futures, it's a bad idea to think in terms of the underlying. Instead, we should model the *forward price* by a multiperiod binomial tree, and consider *options on the forward price* (for example a call, which pays  $(\mathcal{F}_T - K)_+$  at time  $T$ ). Here  $\mathcal{F}$  can be the forward price for delivery at time  $T$ , or it can be the forward price for delivery at some later time  $T' > T$ . We focus as usual on a constant-interest-rate environment; therefore the forward price is equal to the futures price, and an option on the forward price is the same as an option on the futures price.

A key advantage of working with the forward price is that *the interest rate doesn't appear in the formula for the risk-neutral probability  $q$* . Indeed, recall from Section 2 that if  $f_{\text{up}}$  and  $f_{\text{down}}$  are the values of an option at the final states of a one-period binomial tree, then the value of the option at the initial state is

$$f_{\text{now}} = e^{-r\delta t} [q f_{\text{up}} + (1 - q) f_{\text{down}}]$$

where

$$q = \frac{\mathcal{F}_{\text{now}} - \mathcal{F}_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}}.$$

Don't be confused: this is the *same  $q$*  we used when hedging with the underlying. (The interest rate isn't irrelevant: it influences the forward prices, which in turn determine  $q$ .)

To value an option on the forward price, we use the same procedure as earlier in this section: work backwards one timestep at a time, using the single-period binomial valuation formula again and again. To identify the replicating portfolio (or to hedge the option) we use the results obtained the end of Section 2. When using futures, the replicating portfolio holds

$$\alpha = \frac{f_{\text{up}} - f_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}}$$

futures contracts during a timestep when the forward price goes from  $\mathcal{F}_{\text{now}}$  to  $\mathcal{F}_{\text{up}}$  or  $\mathcal{F}_{\text{down}}$  and the option value goes from  $f_{\text{now}}$  to  $f_{\text{up}}$  or  $f_{\text{down}}$ . When using forwards with maturity  $T'$ , the formula is slightly different (if  $r \neq 0$ ): for a subtree with final time  $n\delta t$  the replicating portfolio holds

$$\alpha = e^{r(T' - n\delta t)} \frac{f_{\text{up}} - f_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}}$$

forward contracts.

To see how this works, let's revisit the example done a couple of pages ago: a call with strike 100 on an underlying described by the 3-period binomial tree in Figure 3. As before, we take the interest rate to be  $r = 0$ ; therefore Figure 3 is also the binomial tree of forward prices. The value of the option doesn't change – it is still given by Figure 5. In particular, its initial value is 15.

To replicate the option using futures, we start by going long on  $\alpha = (25 - 5)/(120 - 80) = 1/2$  futures. Our initial portfolio thus consists of this futures position (which has no value) and the 15 we received for the option. Focusing on the first scenario we considered earlier:

**Suppose the forward price goes up to 120.** We receive  $(1/2)(120 - 100) = 10$  due to the change in the futures price; this increases our cash position to 5. The new  $\alpha$  is  $(40 - 10)/(140 - 100) = 3/4$ , so we must acquire another  $1/4$  futures contract. Our portfolio consists of  $3/4$  futures (no value) and 25 cash – matching, as it should, the value of the option.

**Suppose the forward price then goes to 140.** We receive another  $(3/4)(140 - 120) = 15$  due to the change in the futures price; this brings our cash position to 40. The new  $\alpha$  is  $(60 - 20)/(160 - 120) = 1$ , so we acquire  $1/4$  futures contract. Our revised portfolio consists of 1 futures contract and 40 cash – matching, again, the value of the option.

**Suppose the forward price then decreases to 120.** At the final time we must pay the exchange 20 due to the change in the futures price; this brings our cash position to 20. This is, as expected, the payoff of the option.

The same argument works for any scenario: if at each timestep we assume the proper futures position (which depends on which node we occupy at that timestep) then our cash position at maturity exactly matches the payoff of the option. This is because the cash flow on the futures position at each timestep matches the change in the value of the option.

Hedging version of the same calculation: consider the investment bank that sold the option. If it follows the procedure outlined above, then when the option matures the value of its cash position at the end exactly offsets the value of its obligation to the option holder. Thus by pursuing the replicating strategy, the bank can completely eliminate any risk associated with its issuance of the option.

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**A formula for the option price.** Our valuation algorithm is easy to implement. But in the one-period setting we had more than an algorithm: we also had a *formula* for the value of the option, as the discounted expected value using a risk-neutral probability. A similar formula exists in the multiperiod setting. To see this, it is most convenient to work with a general binomial tree. As usual, we focus first on an option on an underlying whose stock price evolution  $s_t$  is described by a binomial tree. Consider, for example, a tree with two time steps. The risk-neutral probabilities  $q_j, 1 - q_j$  are determined by the embedded one-period models. (Remember, the risk-neutral probabilities are characteristic of the market; they don't depend on the contingent claim under consideration.) In this case:

$$q_1 = \frac{e^{r\delta t} s_1 - s_2}{s_3 - s_2}, \quad q_2 = \frac{e^{r\delta t} s_2 - s_4}{s_5 - s_4}, \quad q_3 = \frac{e^{r\delta t} s_3 - s_6}{s_7 - s_6}.$$

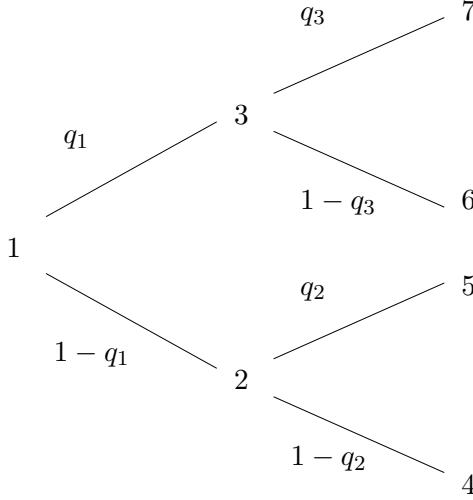


Figure 6: General binomial tree with two time steps.

As we work backward through the tree, we get a formula for the value of the contingent claim at each node, as a discounted weighted average of its values at maturity. In fact, writing  $f(j)$  for the value of the contingent claim  $f$  at node  $j$ ,

$$f(3) = e^{-r\delta t}[q_3 f(7) + (1 - q_3) f(6)]$$

and

$$f(2) = e^{-r\delta t}[q_2 f(5) + (1 - q_2) f(4)]$$

so

$$\begin{aligned} f(1) &= e^{-r\delta t}[q_1 f(3) + (1 - q_1) f(2)] \\ &= e^{-2r\delta t}[q_1 q_3 f(7) + q_1(1 - q_3) f(6) + (1 - q_1) q_2 f(5) + (1 - q_1)(1 - q_2) f(4)]. \end{aligned}$$

It should be clear now what happens, for a binomial tree with any number of time periods:

$$\text{initial value of the claim} = e^{-rN\delta t} \sum_{\text{final states}} [\text{probability of the associated path}] \times [\text{payoff of state}],$$

where the probability of any path is the product of the probabilities of the individual risk-neutral probabilities along it. (Thus: the different risk-neutral probabilities must be treated as if they described independent random variables.)

A similar rule applies to recombinant trees, since they are just special binomial trees in disguise. We must simply be careful to count the paths with proper multiplicities. For example, consider a two-period model with a recombinant tree and  $s_{\text{up}} = us_{\text{now}}$ ,  $s_{\text{down}} = ds_{\text{now}}$ . In this case the formula becomes

$$f(1) = e^{-2r\delta t}[(1 - q)^2 f(4) + 2q(1 - q) f(5) + q^2 f(6)]$$

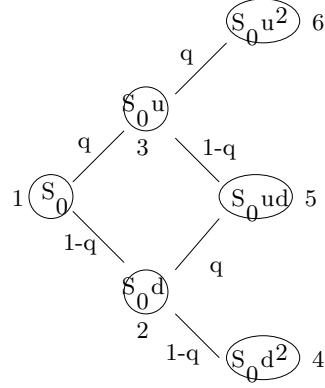


Figure 7: Price states in a multiplicative recombinant tree.

with  $q = (e^{r\delta t} - d)/(u - d)$ , since there are two distinct paths leading to node 5.

The preceding calculation extends easily to recombinant trees with any number of time steps. The result is one of the most famous and important results of the theory: an explicit formula for the value of a European option. This is in a sense the binomial tree version of the Black-Scholes formula. (To really use it, of course, we'll need to know how to specify the parameters  $u$  and  $d$ ; we'll come to that soon.) Consider an  $N$ -step recombinant stock price model with  $s_{\text{up}} = us_{\text{now}}$ ,  $s_{\text{down}} = ds_{\text{now}}$ , and  $s_0$ =initial spot price. Then the present value of an option with payoff  $f(S_T)$  is

$$e^{-rN\delta t} \sum_{j=0}^N \left[ \binom{N}{j} q^j (1-q)^{N-j} f(s_0 u^j d^{N-j}) \right].$$

with  $q = (e^{r\delta t} - d)/(u - d)$ . This holds because there are  $\binom{N}{j}$  different ways of accumulating  $j$  ups and  $N - j$  downs in  $N$  time-steps (just as there are  $\binom{N}{j}$  different ways of getting heads exactly  $j$  times out of  $N$  coin flips.) Making this specific to European puts and calls: a call with strike price  $K$  has present value

$$e^{-rN\delta t} \sum_{j=0}^N \left[ \binom{N}{j} q^j (1-q)^{N-j} (s_0 u^j d^{N-j} - K)_+ \right];$$

a put with strike price  $K$  has present value

$$e^{-rN\delta t} \sum_{j=0}^N \left[ \binom{N}{j} q^j (1-q)^{N-j} (K - s_0 u^j d^{N-j})_+ \right].$$

What changes when we consider an option on the forward price  $\mathcal{F}_t$ , if the forward price evolution is described by a multiperiod binomial tree? Hardly anything! If the stock price tree was multiplicative, then the forward price tree is also multiplicative. The values of  $u$  and  $d$  for the forward price tree are slightly different (if  $r \neq 0$ ) from those of the stock price

tree. But no matter: if  $\mathcal{F}_{\text{up}} = u\mathcal{F}_{\text{now}}$  and  $\mathcal{F}_{\text{down}} = d\mathcal{F}_{\text{now}}$  at each timestep, then (by exactly the same argument as above) an option with payoff  $f(\mathcal{F}_T)$  at time  $T = n\delta t$  has value

$$e^{-rN\delta t} \sum_{j=0}^N \left[ \binom{N}{j} q^j (1-q)^{N-j} f(\mathcal{F}_0 u^j d^{N-j}) \right].$$

at time 0, where  $\mathcal{F}_0$  is the forward price at time 0, and

$$q = \frac{\mathcal{F}_{\text{now}} - \mathcal{F}_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}} = \frac{1-d}{u-d}.$$

Notice that  $0 < q < 1$  exactly if the values of  $u$  and  $d$  (for the forward price tree) satisfy  $d < 1$  and  $u > 1$ . These conditions must hold, for the market to be arbitrage-free.