Derivative Securities – Fall 2007 – Section 10 addendum
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Forwards versus futures. The Section 10 notes include a discussion of *convexity*. It explains, among other things, why the futures price of a bond is a little higher than the forward price. [The version distributed in class said "lower" rather than higher; that was a typo, now corrected in the version on the web.] The following discussion, taken from Section 12.3 of *Quantitative Modeling of Derivative Securities* by Avellaneda and Laurence, provides a more quantitative discussion of this phenomenon.

Another example of convexity discussed in the Section 10 notes is an unconventional forward rate agreement whose payment is made at the beginning of the term rather than at the end. For a more quantitative discussion of that example, showing how Black's formula can be used to value such an instrument, see Hull's Section 27.1 (Application 1).

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Forwards versus futures. There is a well-developed market for futures contracts on treasury bonds. At first this may seem surprising, since there are so many different types of bonds and a futures contract must refer to a well-defined underlying. In practice this difficulty is avoided by rules that permit a variety of similar bonds to be delivered when the contract matures, with cash adjustments depending on the specific bond delivered. This feature makes the futures market complicated and interesting.

Here however I want to focus on a different issue, namely the relationship between forward and future prices. My purpose is partly to emphasize that the two are different, and to get a handle on how the dynamics of interest rates determines forward rates.

The following discussion applies to the binomial-tree-style interest rate model discussed in Section 10 (in which the tree gives the risk-neutral evolution of the short-term interest rate). However we shall use the notation of a continuous-time model (mainly: integrals rather than sums) since this is less cumbersome. Our starting point is the fact that

$$B(t,T)/A(t) = E_{\text{RN}}\left[1/A(T)\right]$$

where $A(t)$ is the value of the money-market fund at time t. In the continuous time setting $A(T) = A(t)\exp\int_t^T r(s)\,ds$ so the preceding formula becomes

$$B(t,T) = E_{\text{RN}}\left[e^{-\int_t^T r(s)\,ds}\right].$$

A typical interest rate future involves 3-month Eurodollar contracts: at the contract's maturity the holder must make a 3-month loan to the counterparty, at interest rate equal to the 3-month-term LIBOR rate. We have called this rate $R(t,T)$, where T=t + 3 months, and t is the maturity date of the futures contract. We know that the associated futures price $\tilde{f}_0(t,T)$ – which determines the daily settlements during the course of the contract – is a martingale under the risk-neutral probabilities, in other words

$$\tilde{f}_0(t,T) = E_{\text{RN}}\left[R(t,T)\right].$$

Let us seek a similar representation for the forward term rate $f_0(t,T)$, defined as above by

$$\frac{1}{1+f_0(t,T)\Delta T} = F_0(t,T) = \frac{B(0,T)}{B(0,t)}$$

with $\Delta T = T - t$. Solving for $f_0(t,T)$ gives

$$f_0(t,T) = \frac{1}{\Delta T \cdot B(0,T)}(B(0,t) - B(0,T)).$$

Rewriting the expression in parentheses as a risk-neutral expectation gives

$$\begin{aligned} f_0(t,T) &= \frac{1}{\Delta T \cdot B(0,T)} E_{\text{RN}}\left[e^{-\int_0^t r(s)\,ds} - e^{-\int_0^T r(s)\,ds}\right] \\ &= \frac{1}{B(0,T)} E_{\text{RN}}\left[e^{-\int_0^t r(s)\,ds} \cdot \frac{1 - e^{-\int_t^T r(s)\,ds}}{\Delta T}\right] \\ &= \frac{1}{B(0,T)} E_{\text{RN}}\left[e^{-\int_0^t r(s)\,ds} \cdot \frac{1 - B(t,T)}{\Delta T}\right], \end{aligned}$$

making use in the last step of the fact that risk-neutral expectations are determined working backward in time. Now, the relation $B(t,T) = 1/[1 + R(t,T)\Delta T]$ can be rewritten as

$$\frac{1 - B(t,T)}{\Delta T} = R(t,T)B(t,T),$$

so we have shown that

$$\begin{aligned} f_0(t,T) &= \frac{1}{B(0,T)} E_{\text{RN}}\left[e^{-\int_0^t r(s)\,ds} R(t,T)B(t,T)\right] \\ &= \frac{1}{B(0,T)} E_{\text{RN}}\left[e^{-\int_0^t r(s)\,ds} R(t,T) e^{-\int_t^T r(s)\,ds}\right], \end{aligned}$$

using once more the fact that risk-neutral expectations are determined working backward in time. Combining the two exponential terms, we conclude finally that

$$f_0(t,T) = \frac{E_{\text{RN}}\left[R(t,T)e^{-\int_0^T r(s)\,ds}\right]}{E_{\text{RN}}\left[e^{-\int_0^T r(s)\,ds}\right]}.$$

Thus the forward rate $f_0(t,T)$ is *not* the risk-neutral expectation of the term rate $R(t,T)$. Rather it is the expectation of $R(t,T)$ with respect to a different probability measure, the one obtained by weighting each path by $\exp\left(-\int_0^T r(s)\,ds\right)$.

It is clear from this calculation that forward rates and futures prices are different. We can also see something about the relation between the two. In fact, writing $R = R(t,T)$ and $D = \exp\left(-\int_0^T r(s)\,ds\right)$ we have

$$\text{forward rate} - \text{futures price} = \frac{E[RD] - E[R]E[D]}{E[D]}.$$

where E represents risk-neutral expectation. If R and D were independent the right hand side would be zero and forward rates would equal futures prices. In general however we should expect R and D to be negatively correlated, since R is a term interest rate and D is a discount factor. Recognizing that $E[RD] - E[R]E[D]$ is the covariance of R and D, we conclude that this expression should normally be negative, implying that

$$\text{forward rate} < \text{futures price}.$$

This is in fact what is observed (the difference is relatively small). A scheme for adjusting the futures price to obtain the forward rate is sometimes called a "convexity adjustment". It should be clear from our analysis that different models of stochastic interest rate dynamics lead to different convexity adjustment rules.