Derivative Securities, Fall 2007 – Homework 4. Distributed 10/17/07, due 10/31/07.

A solution sheet to HW3 will be posted 10/25; a solution sheet to HW4 will be posted 11/8; no late HW's will be accepted once the corresponding solution sheet has been posted.

1. Let F solve the SDE $dF = \mu F\,dt + \sigma F\,dw$, where μ and σ are constant and w is Brownian motion. Find the SDE solved by
   (a) $V(F) = AF$, where A is constant
   (b) $V(F) = \sqrt{F}$
   (c) $V(F) = \cos F$
   (d) $V(F,t) = F^3 t^2$.

2. We continue to assume that F solves $dF = \mu F\,dt + \sigma F\,dw$, where μ and σ are constant and w is Brownian motion. Find a function $V(F)$ such that the process $t \mapsto V(F(t))$ is a martingale (i.e. the SDE describing $V(F)$ has no dt term).

3. This problem should help you understand Ito's formula. If w is Brownian motion, then Ito's formula tells us that $z = w^2$ satisfies the stochastic differential equation $dz = 2w dw + dt$. Let's see this directly:
   (a) Suppose $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$. Show that $w^2(t_{i+1}) - w^2(t_i) = 2w(t_i)(w(t_{i+1}) - w(t_i)) + (w(t_{i+1}) - w(t_i))^2$, whence
   $$w^2(b) - w^2(a) = 2\sum_{i=0}^{N-1} w(t_i)(w(t_{i+1}) - w(t_i)) + \sum_{i=0}^{N-1}(w(t_{i+1}) - w(t_i))^2$$
   (b) Let's assume for simplicity that $t_{i+1} - t_i = (b-a)/N$. Find the mean and variance of $S = \sum_{i=0}^{N-1}(w(t_{i+1}) - w(t_i))^2$.
   (c) Conclude by taking $N \to \infty$ that
   $$w^2(b) - w^2(a) = 2\int_a^b w\,dw + (b-a).$$

4. Here's a cute application of the Ito calculus. Let $w(t)$ be Brownian motion (with $w(0) = 0$), and consider
   $$\beta_k(t) = E[w^k(t)].$$
   Show using Ito's formula that for $k = 2, 3, \ldots$,
   $$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds.$$
   Deduce that $E[w^4(t)] = 3t^2$. What is $E[w^6(t)]$?

   [Comment: the moments of w can also be calculated from its distribution function, since $w(t)$ is Gaussian with mean 0 and variance 1. But the method in this problem is easier, and good practice with Ito's lemma.]

5. Consider the solution of

$$ds = r(t)s\,dt + \sigma(t)s\,dw, \quad s(0) = s_0. \tag{1}$$

where $r(t)$ and $\sigma(t)$ are deterministic functions of time.

(a) $\log s(t)$ is a Gaussian random variable. Show that its mean is $\int_0^t [r(s) - \frac{1}{2}\sigma^2(s)]\,ds$, and its variance $\int_0^t \sigma^2(s)\,ds$.

(b) Show that $s(T) = s_0 \exp\left([\overline{r} - \frac{1}{2}\overline{\sigma}^2]T + \overline{\sigma}\sqrt{T}Z\right)$ where Z is a standard Gaussian,

$$\overline{r} = \frac{1}{T}\int_0^T r(s)\,ds \quad \text{and} \quad \overline{\sigma}^2 = \frac{1}{T}\int_0^T \sigma^2(s)\,ds.$$

[Comment: The price of a non-dividend-paying stock has the form (1) under the risk-neutral measure, if the volatility and interest rate are deterministic functions of time. A forward price also satisfies an equation of this form, with $r(t) \equiv 0$, if the volatility is a deterministic function of time. According to this problem, when the time-dependence of $r(t)$ and $\sigma(t)$ is is known in advance, we can value options by using the standard Black-Scholes formulas with r and σ replaced by $\overline{r}$ and $\overline{\sigma}$.]

6. Suppose a forward price process F has lognormal dynamics with (constant) volatility σ. Assume the interest rate is r (also constant). In HW3 we used risk-neutral valuation to show that the time-0 value of a European option with payoff $F^n(T)$ at time T is

$$e^{-rT} F_0^n \exp\left(\tfrac{1}{2}\sigma^2 T n(n-1)\right).$$

Let's give a different derivation of the same result, using the Black-Scholes PDE for options on forwards.

(a) Substitute $V(F,t) = h(t)F^n$ into the Black-Scholes PDE. What ODE must $h(t)$ solve? What is the appropriate final-time condition?

(b) Verify that $h(t) = e^{-r(T-t)}F^n(t)\exp\left(\frac{1}{2}\sigma^2(T-t)n(n-1)\right)$ solves the ODE you found in (a), with the appropriate final-time condition.