Derivative Securities, Fall 2007 – Homework 3. Distributed 10/3/07, due 10/17/07. *Typo in problem 3 corrected: a squared call has payoff $(F_T - K)^2_+$.*

A solution sheet to HW2 will be posted 10/11; a solution sheet to HW3 will be posted 10/25; no late HW's will be accepted once the corresponding solution sheet has been posted.

Problem 1 provides practice with lognormal statistics. Problems 2-5 explore the consequences of our formula for the value of an option as the discounted risk-neutral expected payoff. Problem 6 makes sure you have access to a numerical tool for playing with the Black-Scholes formula and the associated "Greeks." Problem 7 reinforces the notion of "implied volatility."

Convention: when we say a forward price F_t is "lognormal with drift μ and volatility σ" we mean $\ln F_t - \ln F_s$ is Gaussian with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$ for all $s < t$; here μ and σ are constant. In this problem set we will consider *only* options on a forward price, and we always assume that the forward price is lognormal.

1. Consider a forward price that's lognormal with drift μ and volatility σ. Suppose the forward price now is F_0.

   (a) Give a 95% confidence interval for F_T, using the fact that with 95% confidence, a Gaussian random variable lies within 1.96 standard deviations of its mean.

   (b) Give the mean and variance of F_T.

   (c) Give a formula for the likelihood that call on this forward price with strike price K and maturity T will be in-the-money at maturity.

   (d) If the drift is 16% per annum and the volatility is 30% per annum, what do (a) and (b) tell you about tomorrow.s closing price in terms of today.s closing price?

   (e) What is the probability that $F_T > E[F_T]$? (Note: the answer is not 1/2.)

2. Consider a derivative with payoff F_T^n at maturity. (To be sure there's no confusion: the payoff is F_T raised to the power n.) Show that its value at time 0 is

$$e^{-rT} F_0^n \exp\left[\tfrac{1}{2}\sigma^2 T n(n - 1)\right]$$

   where r is the risk-free rate and σ is the volatility. (Hint: the option's value is $e^{-rT} E_{\text{RN}}[\text{payoff}]$.)

3. Consider a squared call with strike K and maturity T, i.e. an option whose payoff at maturity is $(F_T - K)^2_+$. Give a formula for its value at time 0, analogous to the formula $e^{-rT}[F_0 N(d_1) - KN(d_2)]$ for an ordinary call. (Hint: use the fact that $(e^x - K)^2 = e^{2x} - 2Ke^x + K^2$.)

4. Consider a "cash-or-nothing" option with strike price K, i.e. an option whose payoff at maturity is

$$\begin{cases} 1 & \text{if } F_T \geq K \\ 0 & \text{if } F_T < K. \end{cases}$$

   It can be interpreted as a bet that the forward price will be at least K at time T.

(a) Give a formula for its value at time $t < T$, in terms of the forward price F_t.

(b) Give a formula for its delta. How does the delta behave as t gets close to T?

(c) Why is it difficult, in practice, to hedge such an instrument?

In view of (c) it is not entirely clear that the Black-Scholes valuation formula is valid for such an option. What do you think?

5. Derive all the formulas given in the Section 5 notes for the "Greeks" of calls and puts on a forward price. Make use of the following hints:

(a) In each case, derive the formula for the call option first; then derive the formula for the put option by using put-call parity.

(b) Use the fact that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, which follows by the Fundamental Theorem of Calculus from the definition $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\, du$.

(c) To derive the formulas for the delta and vega of a call, start by showing that $F_0 N'(d_1) = K N'(d_2)$.

6. Suppose r is 5 percent per annum and σ is 20 percent per annum. Let's consider standard put and call options with strike price $K = 50$. Do this problem using the Black-Scholes formulas (not a binomial tree).

(a) Suppose the forward price is $F_0 = 50$ and the maturity is one year. Find the value, delta, vega, and gamma of the put. Same request for the call. Do the same computation for $F_0 = 40$ and $F_0 = 60$.

(b) Graph the value of a European call as a function of the forward price F_0, for maturities of 1, 2 and 3 years. Display all the graphs on a single set of axes, and comment on the trends they reveal.

(c) Same as (b) but for a European put.

[Comment: Use whatever means (matlab, mathematica, spreadsheet) is most convenient, but say briefly what you used. One point of this problem is to visualize the behavior of the pricing formulas. Another is to be sure you have a convenient tool for exploring further on your own.]

7. Find the implied volatility of a 3 year European-style call option if the forward price is presently 100, the strike is 90, the risk-free rate is 3% per annum, and the option price is 21.27. Start with a volatility of 20% and determine the option price and vega predicted by the Black-Scholes theory. Do three iterations of guessing a new volatility based on the previous call price and vega, then re-estimating call price and vega.