Derivative Securities – Homework 3 – distributed 10/11/04, due 10/25/04

Problem 1 provides practice with lognormal statistics. Problems 2-4 explore the consequences of our formula for the value of an option, as the discounted risk-neutral expected payoff. Problem 5 makes sure you have access to a numerical tool for playing with the Black-Scholes formula and the associated "Greeks." Problem 6 examines the question: exactly which binomial trees are consistent, in the continuous-time limit, with our continuous-time valuation formula.

Convention: when we say a risky asset has "lognormal dynamics with drift μ and volatility σ" we mean $\log s(t_2) - \log s(t_1)$ has mean $\mu(t_2 - t_1)$ and variance $\sigma^2(t_2 - t_1)$; here μ and σ are constant. When pricing options, we assume the underlying has lognormal dynamics and pays no dividend, and the risk-free rate is (constant) r.

(1) Consider a stock whose price has lognormal dynamics with drift μ and volatility σ (as defined above). Suppose the stock price now is s_0.

(a) Give a 95% confidence interval for the price at time T, using the fact that with 95% confidence, a Gaussian random variable lies within 1.96 standard deviations of its mean.

(b) Give the mean and variance of the price at time T.

(c) Give a formula for the likelihood that an option with strike price K and maturity T will be in-the-money at maturity.

(d) If the mean return is 16% per annum and the volatility is 30% per annum, what do (a) and (b) tell you about tomorrow's closing price in terms of today's closing price?

(e) What is the probability that $s_T > E[s_T]$? (Note: the answer is not 1/2.)

(2) Consider a derivative with payoff s_T^n at maturity. Show that its value at time t is

$$s_t^n e^{[\frac{1}{2}\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free rate and σ is the volatility of the underlying asset. (Hint: use the option valuation formula $e^{-rT}E_{\mathrm{RN}}[\text{payoff}]$.)

(3) Consider a squared call with strike K and maturity T, i.e. an option whose payoff at maturity is $(s_T - K)_+^2$.

(a) Evaluate its hedge ratio (its "Delta") by differentiating under the integral, then evaluating the resulting expression.

(b) Give a formula for the value of the squared call at time 0, analogous to the standard formula $s_0 N(d_1) - Ke^{-rT}N(d_2)$ for an ordinary call.

(Hint: For part (b) use the fact that $(e^x - K)^2 = e^{2x} - 2Ke^x + K^2$. You could of course differentiate your answer to (b) to find Delta, but that's the hard way.)

(4) Consider a "cash-or-nothing" option with strike price K, i.e. an option whose payoff at maturity is

$$f(s_T) = \begin{cases} 1 & \text{if } s_T \geq K \\ 0 & \text{if } s_T < K \end{cases}$$

It can be interpreted as a bet that the stock will be worth at least K at time T.

(a) Give a formula for its value at time t, in terms of the spot price s_t.

(b) Give a formula for its Delta (i.e. its hedge ratio). How does the Delta behave as t gets close to T?

(c) Why is it difficult, in practice, to hedge such an instrument?

[Comment: Such options are rarely found "naked" but they often arise in "structured products" calling for a fixed payment to be made if an asset price is above a certain value on a certain date. In view of (c) it is not entirely clear that the Black-Scholes valuation formula is valid for such an option. What do you think?)

(5) Suppose r is 5 percent per annum and σ is 20 percent per annum. Let's consider standard put and call options with strike price $K = 50$. Do this problem using the Black-Scholes formulas (not a binomial tree).

(a) Suppose the spot price is $s_0 = 50$ and the maturity is one year. Find the value, Delta, and Vega of the put. Same request for the call.

(b) Graph the value of a European call as a function of the spot price s_0, for several maturities. Display all the graphs on a single set of axes, and comment on the trends they reveal.

(c) Same as (b) but for a European put.

(d) Your answer to (c) should show that the value of the put is lower than $(K - s_0)_+$ for $s_0 < s_*$ and higher for $s_0 > s_*$. Estimate the critical value s_* when the maturity T is 2 years.

[Comment: Use whatever means (matlab, mathematica, spreadsheet) is most convenient, but say briefly what you used. One point of this problem is to visualize the behavior of the Black-Scholes pricing formulas. Another is to be sure you have a convenient tool for exploring further on your own.]

(6) We saw in Section 4 that different binomial trees (associated with different values of μ) can give the same values for options in the continuum limit. So it makes sense to ask: for a given risk-free rate r and volatility σ, which binomial trees give the correct continuum limit? Let us refine this question a bit. We consider only recombinant trees of the form $s_{\text{up}} = us_{\text{now}}$, $s_{\text{down}} = ds_{\text{now}}$. The continuum limit corresponds to $n \to \infty$ time steps of

size $\delta t = T/n$. We expect u and d to depend on n, i.e. $u = u_n$, $d = d_n$. For any fixed n the value of the option is $e^{-rT} E_{\rm RN}[f(s_T)]$; this is the value obtained by working backward through the tree, using the risk-neutral probability $q = q_n = (e^{r\delta t} - d)/(u - d)$. In the continuum limit we know the value should be $e^{-rT} E[f(s_0 e^X)]$ where X is Gaussian with mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Our task is to find conditions on u_n and d_n such that

$$E_{\rm RN}[f(s_T)] \to E[f(s_0 e^X)] \quad \text{as } n \to \infty. \tag{1}$$

The main point of this problem is to show that (1) holds if $u = u_n$ and $d = d_n$ are chosen so that

$$qu + (1-q)d = e^{r\delta t}, \quad qu^2 + (1-q)d^2 = e^{(2r+\sigma^2)\delta t}. \tag{2}$$

Of course the first relation is equivalent to the definition of the risk-neutral probability $q = q_n$, so only the second relation is new. Notice that (2) gives two equations in three unknowns (u, d, q), so there is one remaining degree of freedom.

(a) Define a_n and b_n by $u_n = e^{a_n}$ and $d_n = e^{b_n}$. Show, by arguing as in the Section 4 notes, that (1) holds if

$$n(q_n a_n + (1-q_n) b_n) \to (r - \tfrac{1}{2}\sigma^2)T \quad \text{and} \quad n q_n (1-q_n)(a_n - b_n)^2 \to \sigma^2 T \tag{3}$$

as $n \to \infty$.

(b) Show, by algebraic manipulation, that (2) is equivalent to

$$u = e^{r\delta t}\left(1 + \sqrt{\frac{1-q}{q}(e^{\sigma^2\delta t} - 1)}\right) \qquad d = e^{r\delta t}\left(1 - \sqrt{\frac{q}{1-q}(e^{\sigma^2\delta t} - 1)}\right)$$

so that

$$a_n = r\delta t + \log\left[1 + \sqrt{\frac{1-q_n}{q_n}\omega_n}\right] \qquad b_n = r\delta t + \log\left[1 - \sqrt{\frac{q_n}{1-q_n}\omega_n}\right]$$

with $\omega_n = e^{\sigma^2\delta t} - 1$.

(c) Use the Taylor expansion of $\log(1+x)$ near $x = 0$ to verify the limits (3) as $n \to \infty$.

(d) How should we choose u_n and d_n, if we want $q_n = 1/2$ *exactly* for each n?