Derivative Securities – Homework 1 – distributed 9/13/04, due 9/27/04

(1) We used arbitrage to value a forward contract on a non-dividend-paying asset. Similar principles can be used to value a forward contract on an asset with a dividend yield, or a forward contract for foreign currency (where the foreign interest rate is like a dividend yield).

(a) Suppose the underlying asset pays cash dividends continuously at constant rate q. (This is a good approximation for a stock index fund.) Show that a forward contract with delivery price K and maturity T has present value $S_0 e^{-qT} - K e^{-rT}$, where S_0 is the spot price and r is the risk-free interest rate. What is the forward rate (the choice of K for which the contract has present value 0)?

(b) Now consider a forward contract to buy francs for K dollars/franc at time T. Show that its present value is $S_0 e^{-qT} - K e^{-rT}$ where S_0 is the present exchange rate, r is the risk-free interest rate for dollar investments, and q is the risk-free rate for franc investments. What is the forward exchange rate (the choice of K for which the contract has present value 0)?

(2) [like Jarrow-Turnbull 2.1] The present exchange rate between US dollars and Euros is 1.22 \$/Euro. The price of a domestic 180-day Treasury bill is \$99.48 per \$100 face value. The price of the analogous Euro instrument is 99.46 Euros per 100 Euro face value.

(a) What is the theoretical 180-day forward exchange rate?

(b) Suppose the 180-day forward exchange rate available in the marketplace is 1.21 \$/Euro. This is less than the theoretical forward exchange rate, so an arbitrage is possible. Describe a risk-free strategy for making money in this market. How much does it gain, for a contract size of 100 Euro?

(3) Let $B(t, T)$ be the cost at time t of a risk-free dollar at time T.

(a) Suppose $B(0, 1)$, $B(0, 2)$ and $B(1, 2)$ are all known at time 0 (i.e. interest rates are deterministic). Show that the absence of arbitrage requires $B(0, 1)B(1, 2) = B(0, 2)$.

(b) Now suppose $B(0, 1)$ and $B(0, 2)$ are known at time 0 but $B(1, 2)$ will not be known till time 1. What goes wrong with your argument for (a)? Show that if we know with certainty that $m \leq B(1, 2) \leq M$ then we can still conclude $mB(0, 1) \leq B(0, 2) \leq MB(0, 1)$.

(4) Which functions $\phi(S_T)$ can be the value-at-maturity of a portfolio of calls? Such a portfolio consisting of a_i call options with strike price price K_i, $1 \leq i \leq N$, all having the same maturity date T. (We permit short as well as long positions, i.e. a_i can be positive or negative. We may suppose $0 < K_1 < \ldots < K_N$. The value of this portfolio at maturity is $\phi(S_T) = \sum_{i=1}^{N} a_i (S_T - K_i)_+$.)

(a) Show that ϕ is a continuous, piecewise linear function of S_T, with $\phi(S_T) = 0$ for S_T near 0, and $\phi(S_T) = a_\infty S_T + b_\infty$ when S_T is sufficiently large.

(b) Show that any such ϕ can be realized by a suitable portfolio, and the portfolio is uniquely determined by ϕ. (Hint: think about the graph of ϕ. How does it determine K_i and a_i?)

(c) Show that $a_\infty = \sum_{i=1}^N a_i$ and $b_\infty = -\sum_{i=1}^N a_i K_i$.

(5) An investor holds a European call with strike K_c and maturity T on a non-dividend-paying asset whose current price is S_0. Suppose the investor can write a put with any strike strike price K_p, write a forward with any delivery price K_f, and can borrow any amount B at the risk-free rate (if B is negative this is a loan). What are the conditions on K_p, K_f, and B that make this combination of positions a constructive sale (i.e. that have the same effect as selling the call)?

(6) [like Jarrow-Turnbull 3.10] The present price of a stock is 50. The market value of a European call with strike 47.5 and maturity 180 days is 4.375. The cost of a risk-free dollar 180 days hence is $B(0, 180) = .9948$.

(a) For a European put with a strike price of 47.5 you are quoted a price of 1.450. Show this is inconsistent with put-call parity.

(b) Describe how you can take advantage of this situation, by finding a combination of purchases and sales which provides an instant profit with no liability 180 days from now.

(7) Show, using the absence of arbitrage, that price of a call must be a decreasing function of its strike price. In other words, if the price is $c[S_0, K, T]$ as a function of spot price S_0, strike price K, and maturity T, show that $c[S_0, K_2, T] < c[S_0, K_1, T]$ when $K_1 < K_2$.