Derivative Securities Final Exam

Fall 2004 – G63.2791 – Professor Kohn

- You may bring two 8.5×11 pages of notes [both sides] to this exam.
- Put your answers on the exam paper; use the back of the page if you need more space, and attach additional sheets if necessary. I will grade *only* your exam paper, not your scratch paper.
- Part A consists of 3 "longer-answer" problems, worth 20 points each. Part B consists of 10 "shorter-answer" problems, worth 10 each. The total possible score is thus 160.
- Show your work, and explain all answers (at least briefly). Partial credit will be given for correct ideas.

NAME: ____________________________

A) Longer-answer problems: 20 points each

A1) _______
A2) _______
A3) _______

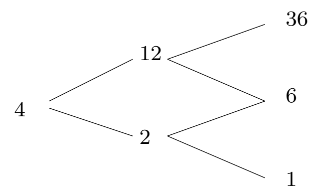
B) Shorter-answer problems: 10 points each

B1) _______	B6) _______
B2) _______	B7) _______
B3) _______	B8) _______
B4) _______	B9) _______
B5) _______	B10) _______

Total: ______________

Part A: Longer-answer questions. Each of the 3 problems in Part A has several parts, worth a total of 20 points.

1. (20 points) Suppose the price of a non-dividend-paying stock is restricted to the multiplicative binomial tree shown below. Notice that $u = 3$, $d = 1/2$. Assume the risk-free



rate satisfies $e^{r\delta t} = 2$. Consider a European put option with strike price $K = 3$.

(a) Find the value of this option by working backward in the tree.

(b) Specify the replicating (hedge) portfolio at time $t = 0$.

(c) Suppose the stock goes up to 12 in the first time period. How should the replicating portfolio be changed?

(d) Now consider the associated American put – with the same strike and underlying, but permitting early exercise. Is its value at time 0 different? Explain briefly.

2. (20 points) Consider a non-dividend-paying stock whose price at time t is $s(t) = e^{y(t)}$ where y solves the stochastic differential equation $dy = \mu y\,dt + \sigma y\,dw$. Notice that $y = \log s$.

   (a) What stochastic differential equation does s solve?

   (b) Consider an option on this stock, and assume its value at time t has the form $V(s(t), t)$. Find the associated hedge, by determining the choice of ϕ that makes $dV - \phi ds$ have no "dw" term.

   (c) What partial differential equation does V solve?

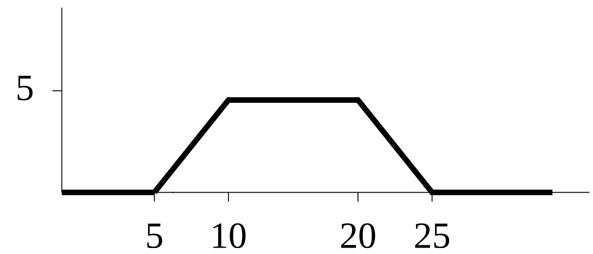
3. Suppose y solves the stochastic differential equation $dy = \mu y\, dt + \sigma y dw$ with $y(0) = y_0$.

(a) Show that if $V(y,t)$ solves $V_t + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy} - rV = 0$ then $e^{-rt}V(y,t)$ is a martingale.

(b) Now suppose that in addition to the PDE, V satisfies the final-time condition $V(y,T) = f(y)$. Show that $V(y_0, 0) = e^{-rT} E_{y(0)=y_0}[f(y(T))]$.

Part B: Shorter-answer questions. The problems in Part B can be answered relatively briefly; they are worth 10 points each.

1. (10 points) Consider the payoff $f(s_T)$ sketched below. How can it be achieved by a combination of call options?



[Note: the payoff is 0 for $s_T < 5$ and for $s_T > 25$; it equals 5 for $10 < s_T < 20$; its slope is 1 for $5 < s_T < 10$; and its slope is -1 for $20 < s_T < 25$.]

2. (10 points) Let $B(0,T)$ be the price in dollars of a risk-free bond worth one dollar at time T, and let $D(0,T)$ be the price in Euros of a risk-free bond worth one Euro at time T. Consider a forward contract, which obligates the holder to buy 100 Euros at K dollars per Euro, at time T. Express its value today (time 0) in terms of the current exchange rate S_0 (in dollars per Euro). Briefly justify your answer.

3. (10 points) Consider the trading strategy that replicates the payoff of a European call on a non-dividend-paying stock with lognormal dynamics. Does it ever require you to take a short position in the underlying? Explain briefly.

4. (10 points) We learned that to hedge an option with value $V(s(t), t)$ you should hold $-\Delta$ units of the underlying, where $\Delta = \partial V / \partial s$. Suppose rather than the underlying, you wish to hedge using futures on the underlying. What should your futures position be at time t? (Assume the risk-free rate is constant r.)

5. (10 points) Consider a non-dividend-paying stock with lognormal dynamics, $ds = \mu s\, dt + \sigma s\, dw$. The risk-free rate is r (assumed constant). Consider the digital option that's worth 1 at time T if $s_T > K$ and 0 otherwise. What is its value at time 0?

6. (10 points) Consider a lognormal stock with continuous dividend yield d. (This means the stock price satisfies $ds = \mu s\, dt + \sigma s dw$, and if you start with one share at time 0 and reinvest all dividends in stock you'll have e^{dt} shares at time t.) Show that early exercise can be optimal for an American call on such a stock.

7. (10 points) Suppose the yield curve is flat at 5%. Consider the following two-year swap: it starts at the end of year 1; the holder pays 6% per annum and receives the floating rate for years 2 and 3; payments are annual. What is the present value of this swap?

8. (10 points) Consider a floorlet with fixed rate R_K for lending one year from now with maturity two years from now. Explain how Black's formula specifies its value now (at time 0) in terms of the notional principal L, the current prices of zero-coupon bonds $B(0, T)$, and a suitable volatility parameter.

9. (10 points) When we use Black's formula to value options with maturity T on an underlying V, we are asserting (or assuming) the existence of a probability measure such that (a) the underlying is lognormal, and (b) the option with payoff $f(V_T)$ has value $B(0,T)E[f(V_T)]$. Show that if these hypotheses hold then the $E[V_T]$ must be the forward price of V for delivery at time T.

10. (10 points) This problem concerns credit risk. Let $B(0,T)$ be the price of a risk-free zero-coupon bond worth T at maturity. Consider two-year corporate bond with principal L, paying annual coupons at 5 percent per annum. Let p_1 be the risk-free probability of default in year 1, and p_2 the risk-free probability of default in year 2. Assume that after default there is no recovery of principal or interest. What is the present value of the corporate bond?