Continuous Time Finance Notes, Spring 2004 – Section 7, *March 10, 2004*
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Brief announcements concerning the rest of this semester: HW 4 will be available after spring break, and will be due March 31. HW5 will be due April 14. HW6 will be due April 28. The final exam will be in the normal class slot on May 5. You may bring two sheets of your own notes (both sides of each page, any font) to the exam, but you may not use books, my notes, HW solutions, etc.

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The March 10 lecture covered one-factor HJM, following closely the treatment in Baxter and Rennie. So I'll just outline what was done in class; please read the book for details.

The essential concept of HJM is to model the evolution of the *entire* term structure, rather than to introduce a specific model for the short rate. The advantage of this framework is that the resulting models are consistent – almost by their very definition – with the initial term structure observed in the market. The main disadvantage is that (except for some special cases, which correspond to short rate models like Hull-White) the HJM framework is difficult to calibrate to market data, and difficult to use for the actual pricing and hedging of instruments. Still, the approach is conceptually attractive, providing a general framework for thinking about interest-based instruments analogous to the familiar diffusion-based framework for thinking about equities. So no modern discussion of interest rates could be complete without touching on this topic.

In Section 5.2 Baxter and Rennie discuss the special case when $\sigma(t,T) = \sigma$ is constant:

$$d_t f(t,T) = \alpha(t,T)\,dt + \sigma\,dw. \tag{1}$$

Integrating the SDE leads easily to explicit formulas for $f(t,T)$ and (setting $T = t$) for $r(t)$. Differentiating the latter one gets the SDE for the short rate

$$dr = \left[\partial_T f(0,t) + \alpha(t,t) + \int_0^t \partial_T \alpha(s,t)\,ds\right] dt + \sigma\,dw. \tag{2}$$

Further integrations give explicit formulas for the values of the money-market fund $B(t) = \exp[\int_0^t r(s)\,ds]$ and the bond $P(t,T) = \exp[-\int_t^T f(t,u)\,du]$. Combining these – and using the fact that

$$\int_0^t \int_0^u \alpha(s,u)\,ds\,du = \int_0^t \int_s^t \alpha(s,u)\,du\,ds$$

(since the 2D region of integration is the triangle in the (s,u) plane where $0 \le s \le u$ and $0 \le u \le t$) we get the explicit formula $P(t,T)/B(t) = e^X$ where

$$X = -\int_0^T f(0,u)\,du - \int_0^t \int_s^T \alpha(s,u)\,du\,ds - \sigma(T-t)w(t) - \sigma\int_0^t w(s)\,ds.$$

Notice that

$$d_t X = \left[-\int_t^T \alpha(t,u)\,du \right] dt - \sigma(T-t)\,dw.$$

By Ito we have $d(e^X) = e^X\,dX + \frac{1}{2}e^X\,dX\,dX$, and this gives

$$d[P(t,T)/B(t)] = [P(t,T)/B(t)] \left\{ \left(\tfrac{1}{2}\sigma^2(T-t)^2 - \int_t^T \alpha(t,u)\,du \right) dt - \sigma(T-t)\,dw \right\}.$$

At this point Baxter and Rennie change to the risk-neutral measure. But it's more transparent in my view to simply assume the original equation holds in the risk-neutral measure. Then the $P(t,T)/B(t)$ must be a martingale, so the drift in the preceding SDE must vanish:

$$\tfrac{1}{2}\sigma^2(T-t)^2 = \int_t^T \alpha(t,u)\,du.$$

This must be true for all maturities T at once. So we can differentiate in T to conclude that

$$\alpha(t,T) = \sigma^2(T-t).$$

In particular: the drift term α in (1) is not something we can choose; it is entirely determined by the volatility.

This special case of HJM is equivalent to the limit $a = 0$ limit of Hull-White (which is know as the Ho-Lee model). To see this, observe that the short rate equation (2) reduces to

$$dr = [\partial_T f(0,t) + \sigma^2 t]\,dt + \sigma\,dw.$$

To see this is the limit of Hull-White when $a \to 0$, recall that Hull-White says

$$dr = (\theta(t) - ar)\,dt + \sigma\,dw$$

with

$$\theta(t) = \partial_T f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

The limit of the right hand side as $a \to 0$ is indeed $\partial_T f(0,t) + \sigma^2 t$ as asserted.

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The situation is not much different for the general one-factor HJM. Start with

$$d_t f(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dw \tag{3}$$

instead of (1). Do all the same integrations as before. Now the integrations involving w cannot be done explicitly; but they don't require any new work, because they're formally analogous to the integrations involving α. Starting exactly as for the simple case done above, we find that

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\,dw(s) + \int_0^t \alpha(s,T)\,dw$$

and

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(s,t)\,dw(s) + \int_0^t \alpha(s,t)\,dw$$

whence the short rate SDE is

$$dr(t) = \left[\partial_T f(0,t) + \int_0^t \partial_T \sigma(s,t)\,dw(s) + \alpha(t,t) + \int_0^t \partial_T \alpha(s,t)\,ds\right] dt + \sigma(t,t)\,dw(t).$$

(Note: usually the drift term in this SDE is not a function of t and $r(t)$ alone. Indeed, even if σ and α are deterministic, the drift term still involves a stochastic integral. Therefore the short rate process is usually non-Markovian.)

Further integrations give once again explicit formulas for the values of the money-market fund $B(t) = \exp[\int_0^t r(s)\,ds]$ and the bond $P(t,T) = \exp[-\int_t^T f(t,u)\,du]$. Combining these – and using the fact that

$$\int_0^t \int_0^u \sigma(s,u)\,dw(s)\,du = \int_0^t \int_s^t \sigma(s,u)\,du\,dw(s)$$

we get the explicit formula $P(t,T)/B(t) = e^X$ where

$$X = -\int_0^T f(0,u)\,du - \int_0^t \int_s^T \alpha(s,u)\,du\,ds - \int_0^t \int_s^T \sigma(s,u)\,du\,dw(s).$$

It is convenient to introduce the notation

$$\Sigma(s,T) = -\int_s^T \sigma(s,u)\,du.$$

Then direct calculation gives

$$d_t X = \left[-\int_t^T \alpha(t,u)\,du\right] + \Sigma(t,T)\,dw(t).$$

By Ito we have $d(e^X) = e^X\,dX + \frac{1}{2}e^X\,dX\,dX$, and this gives

$$d[P(t,T)/B(t)] = [P(t,T)/B(t)]\left\{\left(\tfrac{1}{2}\Sigma^2(t,T) - \int_t^T \alpha(t,u)\,du\right) dt + \Sigma(t,T)\,dw\right\}.$$

Let's assume as before that we are working from the start in the risk-neutral measure. Then P/B must be a martingale, so the drift term must vanish:

$$\tfrac{1}{2}\Sigma^2(t,T) = \int_t^T \alpha(t,u)\,du.$$

This holds for every T, so we may differentiate with respect to T. This gives

$$\alpha(t,T) = \Sigma(t,T)\partial_T \Sigma(t,T).$$

Recalling the definition of Σ, this amounts to the statement that

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)\,du.$$

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How can we get the mean-reverting Hull-White model from HJM? Well, from our analysis of Hull-White, we know that its instantaneous forward rates satisfy

$$d_t f(t,T) = (\text{stuff})\, dt + \sigma e^{-a(T-t)}\, dw$$

under the risk-neutral measure. So we can expect one-factor HJM to specialize to Hull-White when $\sigma(t,T) = \sigma_0 e^{-a(T-t)}$ with σ_0 constant. One of the problems on HW4 asks you to verify this.