Continuous Time Finance, Spring 2004 – Homework 6
Distributed 4/14/04, due 4/28/04

(1) We studied the Dupire equation, for calls on a non-dividend-paying stock. This problem asks you to derive the analogous equation for calls on a foreign currency rate. Since the letter C will be used for the call price, we use S for the foreign currency rate. Recall that under the (domestic investor's) risk-free measure it evolves by

$$dS = (r - q)S\,dt + \sigma(S,t)S\,dw$$

where r is the domestic risk-free rate and q is the foreign risk-free rate. Assume r and q are constant, and assume $\sigma(S,t)$ is a deterministic function of S and t. Let

$$C(K,T) = e^{-rT}E\left[(S_T - K)_+\right]$$

be the time-zero value of a call with strike K and maturity T under this model. Show that it solves

$$C_T = \tfrac{1}{2}K^2\sigma^2(K,T)C_{KK} + (q - r)KC_K - qC$$

for $T > 0$ and $K > 0$, with initial condition

$$C(K,0) = (S_0 - K)_+$$

and boundary condition

$$C(0,T) = e^{-qT}S_0$$

where S_0 is the time-zero spot exchange rate.

(2) Consider scaled Brownian motion with drift and jumps: $dy = \mu dt + \sigma dw + JdN$, starting at $y(0) = 0$. Assume the jump occurences are Poisson with rate λ, and the jump magnitudes J are Gaussian with mean 0 and variance δ^2. Find the probability distribution of the process y at time t. (*Hint*: don't try to solve the forward Kolmogorov PDE. Instead observe that you know, for any n, the probability that n jumps will occur before time t; and after conditioning on the number of jumps, the distribution of y is a Gaussian whose mean and variance are easy to determine. Assemble these ingredients to give the density of y as an infinite sum.) [*Comment*: Using essentially the same idea, Merton gave an explicit formula for the value of an option when y is the logarithm of the stock price under the subjective measure.]