Continuous Time Finance, Spring 2004 – Homework 4
Posted 3/19/04, due 3/31/04

(1) To solve Problem 5 of HW3 you needed to know that if $dr = (\theta - ar)\,dt + \sigma\,dw$ then the function $v(x,t)$ defined by

$$v(x,t) = E_{r(t)=x}\left[e^{-\int_t^T r(s)\,ds} f(r(T))\right] \tag{1}$$

solves

$$v_t + (\theta - ax)v_x + \tfrac{1}{2}\sigma^2 v_{xx} - xv = 0$$

for $t < T$, with final-time condition $v(x,T) = f(x)$. This is a special case of the Feynman-Kac formula. Give a self-contained proof, using the method of HW1, problem 1. (You should assume that the PDE has a unique solution with this final-time condition; your task is to prove that the solution of the PDE satisfies (1).)

(2) The Section 6 notes explain how a trinomial tree can be used to approximate the random walk $dx = \sigma\,dw$, and how working backward in this tree amounts to a standard finite-difference scheme for solving the backward Kolmogorov equation $u_t + \frac{1}{2}\sigma^2 u_{xx} = 0$. Let's try to do something similar for the "geometric Brownian motion with drift" process $dy = \mu y\,dt + \sigma y\,dw$, whose backward Kolmogorov equation is $v_t + \mu y v_y + \frac{1}{2}\sigma^2 y^2 v_{yy} = 0$. Assume the time interval is Δt, and at time $t = n\Delta t$ the tree has nodes at $-n\Delta y, \ldots, n\Delta y$. The process on the tree goes from (y,t) to $(y+\Delta y, t+\Delta t)$ with probability p_u, to $(y, t+\Delta t)$ with probability p_m, and to $(y - \Delta y, t + \Delta t)$ with probability p_d.

(a) How must p_u, p_m, and p_d be chosen to get the means and variances right? What are the conditions for them to be positive?

(b) What is wrong with this scheme?

(3) A better trinomial approximation of "geometric brownian motion with drift" is obtained by recognizing that if $dy = \mu y\,dt + \sigma y\,dw$ then $y = e^z$ with $dz = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dw$.

(a) Consider a trinomial tree process which goes from (z,t) to $(z + \Delta z, t + \Delta t)$ with probability p_u, to $(z, t+\Delta t)$ with probability p_m, and $(z-\Delta z, t+\Delta t)$ with probability p_d. How must p_u, p_m, and p_d be chosen to match the means and variances of the z process? What are the conditions for them to be positive?

(b) Working backward in this tree amounts to a finite-difference scheme for solving the backward Kolmogorov PDE $w_t + (\mu - \frac{1}{2}\sigma^2)w_z + \frac{1}{2}\sigma^2 w_{zz}$ with specified final-time data at $t = T$. In what sense can this also be viewed as a scheme for solving the PDE $v_t + \mu y v_{yy} + \frac{1}{2}\sigma^2 y^2 v_{yy} = 0$?

(Note: The "trinomial tree" scheme for valuing options uses this tree for the z process, with $\mu = r$. However the option value is the *discounted* payoff; this introduces a discount factor of $e^{-r\Delta t}$ at each timestep, and a term $-rw$ in the PDE.)

(4) As we discussed in class, the general one-factor HJM model stipulates

$$d_t f = \alpha(t,T)\, dt + \sigma(t,T)\, dw \tag{2}$$

in the risk-neutral measure. We may choose the volatility $\sigma(t,T)$ arbitrarily, but it determines the drift $\alpha(t,T)$ through the formula

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)\, du. \tag{3}$$

The associated short rate is

$$r(t) = f(0,t) + \int_0^t \sigma(s,t)\, dw(s) + \int_0^t \alpha(s,t)\, ds$$

which solves the SDE

$$dr = \left[\partial_T f(0,t) + \int_0^t \partial_T \sigma(s,t)\, dw(s) + \alpha(t,t) + \int_0^t \partial_T \alpha(s,t)\, ds\right] dt + \sigma(t,t)\, dw(t). \tag{4}$$

Let's verify that when $\sigma(t,T) = \sigma e^{-a(T-t)}$ (with σ constant) we recover the Hull-White model:

(a) Show that $\alpha(t,T) = \frac{\sigma^2}{a} e^{-a(T-t)} \left(1 - e^{-a(T-t)}\right)$.

(b) Show that the SDE (4) reduces in this case to $dr = (\theta(t) - ar)\, dt + \sigma\, dw$ with

$$\theta(t) = \partial_T f(0,t) + a f(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

(5) This problem revisits HW3, problem 1, using the general one-factor HJM theory $d_t f(t,T) = \alpha(t,T)\, dt + \sigma(t,T)\, dw$ rather than Vasicek-Hull-White. Well, not the most general theory: you must assume for this problem that $\sigma(t,T)$ is a given, deterministic function of t and T (whereas the general HJM framework permits it to be random, provided it depends only on time-t information). Besides the formulas (2)-(3), you'll need the fact that

$$d_t[P(t,T)/B_t] = [P(t,T)/B_t]\Sigma(t,T)\, dw \tag{5}$$

where B_t is the money-market account and

$$\Sigma(t,T) = -\int_t^T \sigma(t,u)\, du.$$

(a) Show that for $t \le \tau \le T \le S$, the random variable $\ln[P(\tau,S)/P(\tau,T)]$ is normal under the risk-neutral measure, and its variance (given information at time t) is

$$\int_t^\tau (\Sigma(u,S) - \Sigma(u,T))^2\, du.$$

(b) To apply Black's formula, we need the statistics of $\ln[P(\tau, S)/P(\tau, T)]$ under the forward measure, not the risk-neutral measure. (The forward measure is the one for which $V_t/P(t,T)$ is a martingale whenever V_t is the value of a tradeable.) Show that if w is Brownian motion under the risk-neutral measure and $\overline{w}$ is Brownian motion under the forward measure then

$$d\overline{w} = -\Sigma(t,T)\,dt + dw.$$

(Hint: specialize the calculation on page 9 of the Section 4 notes to the case at hand.)

(c) Use the result of (b) to show that $\ln[P(\tau, S)/P(\tau, T)]$ is also normal under the forward measure, and its variance is the *same* under the forward and risk-neutral measures.

(d) Consider a call option with maturity T and strike K, on a zero-coupon bond with maturity $S > T$. Its payoff at time T is $(P(T,S) - K)_+$. Show that its value at time t is

$$P(t,S)N(d_1) - KP(t,T)N(d_2)$$

where

$$d_1 = \frac{\ln[\frac{P(t,S)}{P(t,T)K}] + \frac{1}{2}s^2}{s}, \quad d_2 = d_1 - s$$

where s is defined by

$$s^2 = \int_t^T (\Sigma(u,S) - \Sigma(u,T))^2\,du.$$

(6) This problem revisits HW3, problem 2, using the general one-factor HJM theory. Consider the call option valued in problem 5.

(a) What trading strategy produces a replicating portfolio using tradeables $P(t,S)$ and $P(t,T)$?

(b) What trading strategy produces a replicating portfolio using tradeables $P(t,S)$ and the money market fund B_t?

(c) What trading strategy produces a replicating portfolio using two bonds $P(t,T_1)$ and $P(t,T_2)$, where T_1 and T_2 are arbitrary (distinct) values greater than T?