Continuous Time Finance, Spring 2004 – Homework 3
Distributed 2/27/04, due 3/10/04

(1) Assume the Vasicek model $dr = (\theta - ar)\,dt + \sigma\,dw$ for the risk-neutral short rate process. Consider a call option with maturity T and strike K, on a zero-coupon bond with maturity $S > T$. Its payoff at time T is $(P(T,S) - K)_+$. Show using Black's formula that the value of this option at time t is

$$P(t,S)N(d_1) - KP(t,T)N(d_2)$$

where

$$d_1 = \frac{1}{\sigma_p}\log\frac{P(t,S)}{P(t,T)K} + \tfrac{1}{2}\sigma_p, \quad d_2 = \frac{1}{\sigma_p}\log\frac{P(t,S)}{P(t,T)K} - \tfrac{1}{2}\sigma_p$$

with

$$\sigma_p = \sigma\left(\frac{1 - e^{-2a(T-t)}}{2a}\right)^{1/2} B(T,S).$$

(The function $B(T,S)$ is the one from the representation $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$.)

(2) Since Vasicek is a one-factor model, the call option of Problem 1 can be replicated by a self-financing trading strategy using any pair of tradeables.

(a) What trading strategy produces a replicating portfolio using tradeables $P(t,T)$ and $P(t,S)$?

(b) What trading strategy produces a replicating portfolio using tradeables $P(t,S)$ and the money market fund $B(t)$?

(c) What trading strategy produces a replicating portfolio using two bonds $P(t,T_1)$ and $P(t,T_2)$, where T_1 and T_2 are arbitrary (distinct) times greater than T?

(3) The call option of Problem 1 can be viewed as an exchange option involving the zero-coupon bonds $S_1 = KP(t,T)$ and $S_2 = P(t,S)$; indeed, its payoff at time T is $(S_2 - S_1)_+$. We discussed exchange options in Section 3. Can the method we applied to exchange options be used to solve Problem 1? Explain.

(4) Suppose instead of Vasicek we use Hull-White for the short-rate: $dr = (\theta(t) - ar)\,dt + \sigma\,dw$.

(a) Consider a call on a zero-coupon bond, as in Problem 1. Show that the valuation formula given in Problem 1 remains valid.

(b) Does this mean that the value of the option doesn't depend on $\theta(t)$? Explain.

(5) Suppose we didn't know the answer to Problem 1, and we decided to value the call by solving a PDE instead of by using Black's formula. What PDE would we have to solve? (Be sure to specify the final-time condition). How does the solution determine the value of the option? (Note that by solving Problem 1, we have given a formula for the solution of this PDE!)