Continuous Time Finance, Spring 2004 – Homework 2
Distributed 2/4/04, due 2/18/04

(1) Consider a market with one source of randomness, a scalar Brownian motion w. Suppose the price S of a stock satisfies

$$dS = \mu(S,t)S\,dt + \sigma(S,t)S\,dw$$

where μ and σ are known functions of S and t. Let r be the risk-free rate, assumed constant for simplicity. We know that all tradeables must have the same market price of risk

$$\lambda = \frac{\mu - r}{\sigma}.$$

Let's check this for the special case of a European option on S with maturity T and payoff $f(S_T)$. Use the Black-Scholes PDE to verify that this option's market price of risk is the same as that of the underlying.

(2) Let S_1 and S_2 be stocks with constant drift and volatility

$$dS_1 = \mu_1 S\,dt + \sigma_1 S\,dw_1, \quad dS_2 = \mu_2 S\,dt + \sigma_2 S\,dw_2,$$

and assume that w_1 and w_2 have (constant) correlation ρ. The risk-free rate is r (also constant). The Section 3 notes discuss the pricing of an exchange option with payoff $(S_2(T) - S_1(T))_+$. Find the trading strategy that replicates this payoff. (Give your answers in terms of cumulative normal distribution functions.)

(3) Now let's consider the same two stocks as in Problem 2, but an option with a general payoff $f(S_1(T), S_2(T))$.

(a) Identify the risk-neutral SDE for S_1 and S_2.

(b) Show that the value of the option at time t is $V(S_1(t), S_2(t), t)$ where $V(x, y, t)$ solves

$$V_t + rxV_x + ryV_y + \tfrac{1}{2}\sigma_1^2 x^2 V_{xx} + \tfrac{1}{2}\sigma_2^2 y^2 V_{yy} + \rho\sigma_1\sigma_2 xy V_{xy} - rV = 0$$

with final-time condition

$$V(x, y, T) = f(S_1, S_2, T).$$

(c) Show that if the payoff has the special form $g(S_2/S_1)S_1$ it suffices to solve the simpler, one-space-dimensional PDE $U_t + \frac{1}{2}\hat{\sigma}^2 x^2 U_{xx} = 0$ with final-time condition $U(x,T) = g(x)$. (Here, as in Section 3, $\hat{\sigma} = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)^{1/2}$.) How does U determine the value of the option?

(4) Let S be the price of a stock in pounds, and let C be the exchange rate in dollars per pound. Assume

$$dS = \mu_S S\,dt + \sigma_S S\,dw_S, \quad dC = \mu_C C\,dt + \sigma_C C\,dw_C,$$

where the drifts and volatilities are constant. Let r and u be the risk-free rates in dollars and pounds respectively, also constant. Consider a quanto call, whose payoff in dollars is $(S_T - K)_+$. The Section 3 notes discuss how to price it. Find the trading strategy that replicates it. (Give your answers in terms of cumulative normal distribution functions.)

(5) For the same stock and exchange rate processes as in Problem 4, consider a general quanto whose payoff in dollars is $f(s_T)$. What PDE should you solve to price it? How does the solution of the PDE determine a dollar investor's replicating portfolio?