Continuous Time Finance, Spring 2004 – Homework 1
Distributed 1/28/04, due 2/4/04

(1) In the Section 1 notes, we proved that if V solves the Black-Scholes PDE with final-value f, then $V(S_0, 0) = e^{-rT} E[f(S_T)]$ where S solves the SDE $dS = rS\,dt + \sigma S\,dw$ with initial value $S(0) = S_0$. Let's do something similar for a stochastic interest rate. Suppose the spot rate r_t solves a diffusion of the form $dr = \alpha\,dt + \beta\,dw$ with $r(0) = r_0$, where $\alpha = \alpha(r,t)$ and $\beta = \beta(r,t)$ are fixed functions of r and t. Consider the function $U(r,t)$ defined by solving $U_t + \alpha U_r + \frac{1}{2}\beta^2 U_{rr} - rU = 0$ with final value $U(r,T) = 1$. Show that

$$U(r_0, 0) = E\left[e^{-\int_0^T r(s)\,ds}\right].$$

[Comment: if the SDE for r is the risk-neutral process, then $U(r_0, 0)$ is the value of a zero-coupon bond that pays one dollar at time T. Hint: show that $U(r(t),t)\exp\left(-\int_0^t r(s)\,ds\right)$ is a martingale.]

(2) Consider a non-dividend-paying stock whose share price satisfies $dS = \mu S\,dt + \sigma S\,dw$, and assume for simplicity that the risk-free rate r is constant. Consider a European option with maturity T and payoff $f(S_T)$. We now have two apparently different ways to price and hedge it:

(a) Using the Black-Scholes PDE. The value at time t is $V(S_t, t)$ where V solves the Black-Scholes PDE with final-value f, and the hedge portfolio consists of $\phi_t = \frac{\partial V}{\partial S}(S_t, t)$ stock and $(V(S_t,t) - \phi_t S_t)/B_t$ units of the risk-free bond whose value at time t is B_t.

(b) Using the Girsanov's theorem and the martingale representation theorem. This means we must find a risk-neutral measure Q with respect to which S_t/B_t is a martingales; then the option value at time t is $V_t = B_t E_Q[f(S_T)/B_T \,|\, \mathcal{F}_t]$, and the hedge ratio ϕ_t is determined by the martingale representation theorem, which tells us that $d(V/B) = \phi_t\,d(S/B)$ for some ϕ_t.

Show these two frameworks are consistent. In other words, show that the value and hedge defined by (a) satisfy the properties asserted by (b).

(3) Consider the discussion in the Section 2 notes concerning options on foreign exchange rates.

(a) What PDE should the dollar investor solve to value an option with payoff $f(C_T)$? How does it determine the hedge portfolio?

(b) What PDE does the pound investor solve to value the same option? How is his hedge portfolio related to that of the dollar investor?

(c) Use these results to give another proof that the two investors price the option consistently.