Calculus of Variations, Spring 2017
MATH-GA 2660.003 (Advanced Topics in Analysis)
Robert V. Kohn, kohn@cims.nyu.edu
Mondays 1:25-3:15, WWH 312

Prerequisites: Real Variables and a first course in PDE. (PDE I is a start, but is not really enough; simultaneous registration in PDE II is sufficient.)

Calendar: First class 1/23. No class 2/20 (Presidents' Day) and 3/13 (Spring Break). Last class 5/8.

Course Description: A modern introduction to the Calculus of Variations, with equal emphasis on theory and applications. Topics will include: existence of solutions and convergence of numerical schemes; convex duality; one-dimensional variational problems; multidimensional nonconvex problems; relaxation; Gamma-convergence; homogenization; and energy-driven pattern formation. Along the way, we'll discuss many applications including minimal surfaces, optimal control, nonlinear elasticity, composite materials, and pattern formation problems involving defects or walls.

All lectures will be supported by (handwritten) lecture notes, which will include selected exercises (recommended but not required) and suggestions where to read more.

The main course requirement, for registered students, is to give a 30-minute presentation near the end of the semester. The topic can be an extension of material covered in the lectures, or some other aspect of the calculus of variations (eg something close to the student's research interests). There will be no final exam.

Website: Lecture notes will be posted on my website (see the *teaching* link at math.nyu.edu/faculty/kohn). I'll also use an NYU-Classes site, mainly for communication by email, and for posting material meant only for students in the class. Non-registered auditors can ask to be added to the NYU-Classes site.

Office hours: My Spring 2017 office hours are Thurs 4-6 (WWH 502).

Tentative semester plan:

- The direct method of the calculus of variations (1/23)
- Convex duality and applications (1/30, 2/6, 2/13)
- One-dimensional variational problems (2/27, 3/6)
- Optimal control (3/20)
- PDE-constrained optimization: sensitivity analysis via the adjoint equation (3/27)
- Some motivating multidimensional examples: elasticity, optimal design, and martensitic phase transformation (4/3)

- Relaxation and quasiconvexity (4/10, 4/17)
- A first example of Gamma-convergence: the Modica-Mortola problem (4/24)
- A second example of Gamma-convergence: homogenization (5/1)
- Singular perturbation and the length scale of microstructure (5/8)

This outline omits many important themes in the Calculus of Variations, including the mountain pass lemma, regularity of minimizers or critical points, harmonic maps, and Ginzburg-Landau vortices. One semester is short!

Some books: The following sources will be useful to students who want to reinforce and/or go beyond the material covered in these lectures. All will be placed on reserve; a few (as noted) are also available electronically through Bobcat.

(1) B. Dacorogna, *Introduction to the Calculus of Variations*, Imperial College Press, 2008. Starts at a fairly basic level, like this class – but much narrower in scope than this class.

(2) Kevin W. Cassel, *Variational Methods with Applications in Science and Engineering*, Cambridge Univ Press, available electronically. Another fairly basic introduction. Compared to item (1) this one is broader, with many more examples and applications (and somewhat less theory).

(3) I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North Holland, 1976. Our treatment of convexity and duality will more concrete and more basic than what's in this book. But Ekeland and Temam is a place to read more about applications of those techniques to PDE problems.

(4) J. Jost and X. Li-Jost, *Calculus of Variations*, Cambridge Univ Press. My discussion of the one-dimensional calculus of variations will follow the early sections of this book.

(5) G. Buttazzo, M. Giaquinta, and S. Hildebrandt, *One Dimensional Variational Problems: an Introduction*, Oxford University Press. More extensive than Jost and Li-Jost (I will draw some material from here as well).

(6) L. Hocking, *Optimal Control: an Introduction to the Theory, with Applications*, Oxford Univ Press 1991. A readable, example-oriented treatment of optimal control. My treatment will be different (Hocking emphasizes the Pontryagin maximum principle, while I'll emphasize Hamilton-Jacobi equations), so Hocking provides a nice complement.

(7) L.C. Evans, *Partial Differential Equations*, Amer Math Society. Covers basic PDE tools that we'll be using (eg existence of minimizers by the direct method), and also many PDE aspects of the calculus of variations that we won't have time to address (including the link between optimal control and viscosity solutions of Hamilton-Jacobi equations).

(8) B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, 2nd edition, 2008, available electronically. A good place to read about quasiconvexity and relaxation.

(9) A. Braides, *Gamma-Convergence for Beginners*, Oxford University Press 2002, available electronically. A good source for examples of Gamma-convergence, though its focus on mostly 1D problems is somewhat limiting.

(10) D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Oxford Univ Press. A good place to learn about homogenization of 2nd order linear elliptic PDE.

These books do not cover everything we'll do; some segments are best supported by articles rather than books. References to relevant articles will be provided as appropriate.