Student presentations for Advanced Topics in Analysis: Calculus of Variations

As announced on the syllabus, each registered student must give a 30-minute presentation at the end of the semester. My tentative plan is that 1/3 of the presentations will be Monday 5/8 (the last official class day), 1/3 on Tues 5/9 (reading day), and 1/3 on Monday 5/15 (the exam-week class slot).

Projected presentations are permitted, but I suggest a blackboard presentation as the default. If two students want to prepare coordinated presentations (on a topic broad enough that it deserves 60 minutes) that is OK.

The deadline for choosing a topic for your presentation is *Friday March 31.* But I encourage you not to wait – choosing earlier means you'll have more time to prepare. When you have a tentative topic, please (a) send me a short email describing it, and (b) come see me (by making an appointment or by stopping by during my Thurs 4-6 office hours) to briefly discuss what you have in mind.

Some ideas linked to Lectures 1-4 are given below. (Lectures 6 and 7 will be on optimal control; there are definitely possibilities related to that material as well. If you'd like some suggestions of that type, feel free to ask.)

You are also welcome to choose something quite separate from the topics we're covering in lecture. For example: you may choose a topic related to your research interests, or a classic topic we won't be covering (such as the Mountain Pass Lemma as a tool for proving existence of critical points).

Some candidate topics for presentations related to Lectures 1-4

(1) *The Lavrentiev phenomenon.* The Lecture 1 notes mention (though we skipped this in class) that for the variational problem $\inf_{u(0)=0,\, u(1)=1} \int_0^1 (u^3 - x)^2 u_x^6 \, dx$ the obvious minimizer is $u(x) = x^{1/3}$, however *if u is restricted to the class of Lipschitz continuous functions then the minimum value is bounded away from* 0. Thus: for some classes of problems (including this one), the minimum value depends on the choice what class of functions are admissible. This is known as the "Lavrentiev phenomenon." Some candidate presentation ideas in this direction:

   (a) The example just mentioned is discussed in Section 4.3 of the book of Buttazzo-Giaquinta-Hildebrandt. Read and present what's there. Perhaps also discuss conditions assuring this phenomenon does *not* occur.

   (b) The paper "On Lavrentiev's Phenomenon" by V.V. Zhikov (Russian Journal of Mathematical Physics vol 3, no 2, 1995, 249-269) discusses some multidimensional examples of the Lavrentiev phenomenon, for example of the form $\min_{u=\phi \text{ at } \partial\Omega} \int_\Omega |\nabla u|^{p(x)}$ where the exponent $p(x)$ is a suitable piecewise constant function of x. Read and present one (or perhaps a few) of the examples in this paper. (The Zhikov paper may be hard to find online; feel free to ask me for a copy.)

(2) *Convex duality for some L^1 optimization problems.* Convex duality is a broad subject, rich with opportunities for presentations. Here are two examples related to material we discussed in Lecture 4:

(a) In the total-variation-based approach to image denoising, the main idea is to solve $\min_u \int_\Omega |u-g|^2\,dx + \lambda \int_\Omega |\nabla u|\,dx$, where $g(x)$ (representing the noise-corrupted image) is given. A recent paper by M Zhu, S Wright, and T Chan, "Duality-based algorithms for total-variation-regularized image restoration" (Computational Optimization and Applications 47(3), 2010, 377-400) discusses some approaches to its solution that make use of duality.

(b) In multivariate statistics, Lasso regression amounts to the problem $\min_{\beta \in R^p} \|y - A\beta\|_2^2 + \lambda\|\beta\|_1$, where $y \in R^n$ is the data and A is an $n \times p$ matrix. (The interesting case is $p > \text{rank}(n)$, when the unregularized problem is underdetermined. Note that the first (regression) term uses the L^2 norm, while the second (regularization) term uses the L^1 norm. (Lasso stands for "least absolute shrinkage and selection operator.") This problem's dual is discussed and exploited in a paper by M Osborne, B Presnell, and B Turlach, "On the lasso and its dual" (Journal of Computational and Graphical Statistics 9(2), 319-337, 2000).

(3) *Algorithms for L^1 optimization.* You can't easily use Newton's method – nor even steepest descent – on an L^1 optimization. The "split Bregman" method works pretty well for problems like total-variation-based image denoising. One treatment is: T Goldstein, S Osher, "The split Bregman method for L1-regularized problems," (SIAM J Imag Sci 2(2), 2009, 323-343).

(4) *Stability of the Simons' cone.* We used the calibration method to show that the Simons' cone is the unique area-minimizing hypersurface surface in B_R, given its boundary condition at ∂B_R. Actually, more is true: if a hypersurface with the same boundary has nearly the area of the Simons' cone, then it must be close to the Simons' cone. Moreover similar results are true for other minimal cones as well (the so-called Lawson cones). These (and related) results are the focus of G De Philippis and F. Maggi, "Sharp stability inequalities for the Plateau problem" (J Diff Geom 96(3), 2014, 399-456). An outline of the main ideas would be a good presentation topic.