Calculus of Variations, Lecture 9, 4/3/2017

Note: I just became aware of a set of lecture notes by Filip Rindler (U of Warwick) that's pretty close to the level + spirit of this class. They will be published soon as a book (Springer), but for now they are online at

www2.warwick.ac.uk/fac/sci/maths/people/staff/filip_rindler/ma496-17_ln.pdf

(or by googling the title, "Introduction to the Modern Calculus of Variations).

We have, up to now, focused largely on var'l plms that have solutions, though we know from simple examples that lots of var'l plms don't have solutions (for example: $\min \int_0^1 (u_x^2-1)^2 + u^2\,dx$)

Today: I'll discuss some examples from applications of var'l problems that "don't have solutions" for reasons that are physically quite natural. As in the 1D example (above), minimization requires "microstructure", and the natural goals are to

a) identify the favorable microstructures

b) give an algorithm for constructing minimizing sequences

We'll achieve both (in some examples) by considering a suitable <u>relaxed</u> variational problem.

<u>Example 1: thin elastic sheets</u> (reference: AC Pipkin, IMA J Appl Math 36, 1986, 85-99; see also my 2014 PCMI lectures, on my website, esp Lecture 2)

Consider a <u>thin rubber sheet</u>. To keep things simple, let's use the <u>incompressible</u> <u>neoHookean</u> elastic energy for 3D elasticity, ie

$$W_{3D}(F) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3$$

where $\{\lambda_i\}$ are the prin stretches (eigenvalues of $(F^TF)^{1/2}$) and $\lambda_1\lambda_2\lambda_3 = 1$ is a constraint.

For a <u>thin sheet</u> it's natural to suppose the deformation gradient is constant wrt thickness. So we can focus on the map $u : \mathbb{R}^2 \to \mathbb{R}^3$ taking the (flat) reference sheet to its position in $\mathbb{R}^3$. Since $|Du \cdot v|^2 = \langle (Du)^T Du\, v, v\rangle$ for $v \in \mathbb{R}^2$, stretching + compression in-plane are assoc to the eigenvalues of the 2×2 matrix $(Du^T Du)^{1/2}$; call these λ_1, λ_2. By incompressibility of the rubber, the energy (per unit thickness) is thus

$$W_{2D}(Du) = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} - 3$$

The var'l pblm

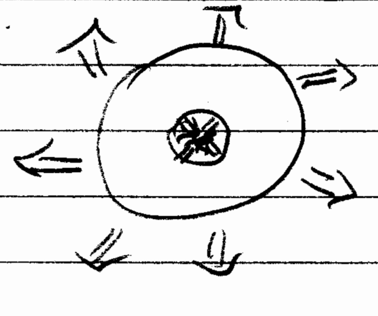
$$\int_\Omega W_{2D}(Du)\, dx$$

(with $u = u_0$ at $\partial\Omega$ if the deformation is fixed at the bdry, or an additional bdry term if a load is applied at the bdry) seems to describe the sheet. But it is "in need of relaxation" in the sense that

a) $\int_\Omega W_{2D}(Du)\, dx$ is not lower semicont's

b) for some choices of loads or bdry conds the min is not achieved!

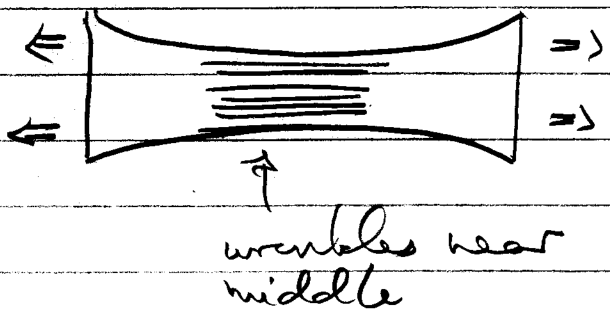
A specific instance of (b): let Ω = a planar annulus, and suppose uniform loads are applied at the inner + outer edges (in the radial dirn)



Then the inner part of the annulus needs to <u>wrinkle</u> to avoid compression in the θ direction (see my 2014 CPAM paper with Peter Bella).

Other examples of wrinkling:

- a rectangular sheet, pulled by a stiff ruler at 2 opposite sides

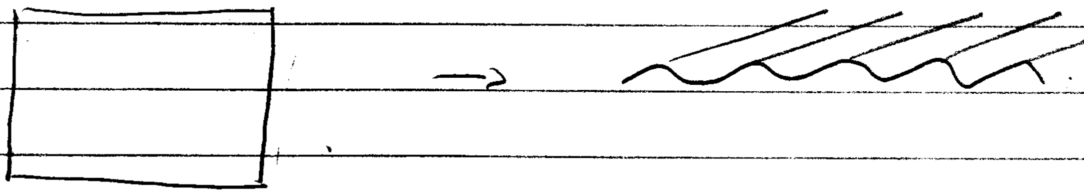


(see Cerda + Mahadevan, Phys Rev Lett 90, 2003)

- the wrinkles one sees near the seams of a mylar balloon when it's blown up (for an effort to design balloons without such wrinkles — using the relaxed energy among other tools — see M Skouras et al, "Designing inflatable structures," ACM Siggraph 2014

What is going on mathematically? Well, W_{2D} prefers $\lambda_1 = \lambda_2 = 1$. But it is nonconvex and not lower semicontinuous since overall compression can be achieved by wrinkling. For example if we consider

$$u_\varepsilon(x) = (u_1, u_2, u_3)(x_1, x_2) = \left(\alpha X_1 + \varepsilon \varphi_1\left(\tfrac{X_1}{\varepsilon}\right), \varepsilon \varphi_2\left(\tfrac{X_1}{\varepsilon}\right), X_2\right)$$



with (φ_1, φ_2) a periodic fn of 1 variable s.t.

$$(\alpha + \varphi_1'(t))^2 + (\varphi_2'(t))^2 = 1$$

(which is possible when $0 \le \alpha < 1$) we see that u_ε is an isometry (so $\lambda_1 = \lambda_2 = 1$ and $W_{2D}(Du_\varepsilon) = 0$) but u_ε converges to

$$u_0(x) = (\alpha X_1, 0, X_2)$$

which is not an isometry (so $W_{2D}(Du_0) > 0$).

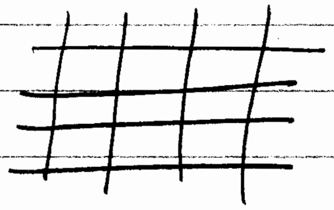
Example 2: Twinning due to martensitic phase transformation (Reference: Ball + James, Phil

Trans Royal Soc London A, vol 338 (1992) 389-450; or, for more context, the book "Microstructure of Martensite" by K. Bhattacharya (Oxford Univ Press, 2004)

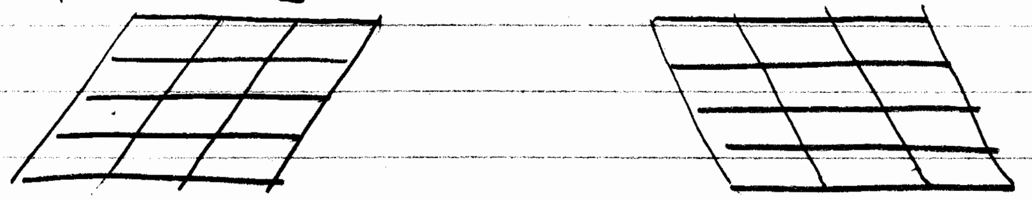
The real phenomenon here is 3D, but I'll explain it in 2D using a "square-to-parallelogram" phase transition

A "shape-memory material" is a crystalline material with a single (high-symmetry) structure at high temp ($T > T_c$) and several (symmetry-related) possible structures at lower temps ($T < T_c$).

2D version: cubic at $T > T_c$



two phases for $T < T_c$ (each with unit cell = parallelogram).



(Both are sheared relative to the cubic phase)

If we ignore issue of nucleation + motion of phase bdries, we can suppose the material chooses the phase of lowest energy for a given defn gradient. So the elastic energy is

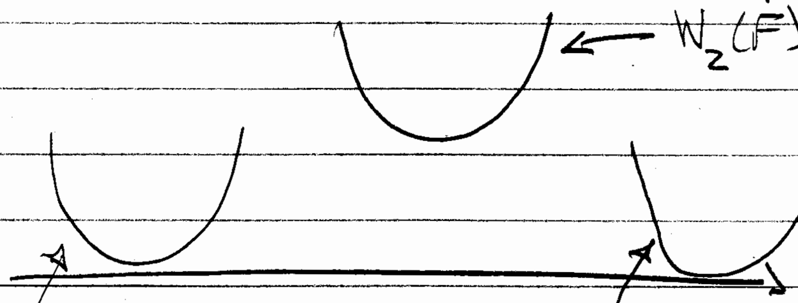
$$\int_{\Omega} W(F)\,dx \qquad F = Du \text{, the } 3\times 3 \text{ def gradient}$$

where

$$W = \min_j \{ W_j(F) \}$$

with W_j = elastic energy density of j^{th} phase.

Picture at $T < T_c$, for the 2D model problem:



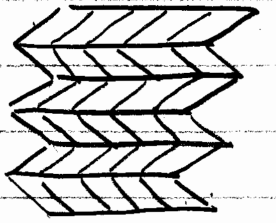
$W_2(F)$ prefers $F \in SO(2)$

$W_1(F)$ prefers $F = R\begin{bmatrix} 1 & -\delta \\ 0 & 1 \end{bmatrix}$ with $R \in SO(2)$

$W_3(F)$ prefers $F = R\begin{bmatrix} 1 & +\delta \\ 0 & 1 \end{bmatrix}$ with $R \in SO(2)$

(each W_j must be frame-indifferent, of course).

The sheared phases can be mixed in layers:



"atomic scale picture" assoc piecewise linear, cont's def $u(x)$ st $F = Du$ takes just 2 values, $\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & -\delta \\ 0 & 1 \end{bmatrix}$

As the vol fractions vary, avg F can be $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ for any $|\lambda| < \delta$.

Actually: The sheared phases can be mixed in layers with two distinct possible normals. In fact, if $U_+ = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$, $U_- = \begin{pmatrix} 1 & -\delta \\ 0 & 1 \end{pmatrix}$, layering requires

$$R_+ U_+ - R_- U_- = a \otimes n$$

One soln was obvious ($R_+ = R_- = I$), but there is another. (Sketch: replacing a by $R_+ a$ and R_- by $R_+ R_-$, we may assume wlog that $R_+ = I$, so we're left looking for θ st

$$\det \left[U_+ - \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} U_- \right] = 0.$$

A little algebra reveals that there are two solutions.)

Consequence: below T_c, the material has lots of zero-energy configurations. But if you heat it above T_c it springs back to the original configuration (since the lattice returns to the cubic phase). Thus the name "shape memory".

Wrapping around these examples:

a) when there's a failure of lsc it can often be detected by examples with a one-dimensional or layered character. [But not always, since we know from Sverak's work that rank-one convexity is not equivalent to quasiconvexity]

b) applications sometimes lead us to consider energies that are (very) nonconvex, and definitely not lower semicontinuous.

c) natural goal in such settings is to understand the energetically-preferred microstructures (equivalently: to identify energy-minimizing sequences)

d) in most settings there is a natural "regularization" that restores existence. (For thin elastic sheets: bending energy; for martensitic phase transf: interfacial energy). Understanding the action of the regularizing term (eg: what is the length scale of the "microstructure") becomes a natural focus of attention.