Calculus of Variations, Lecture 8, 3/27/2017

We return now to the theme of Lecture 1: the "direct method" in the calculus of var'ns.

Recall from Lecture 1: in considering $\int_\Omega W(Du)$, the "direct method" starts with a minimizing sequence; if growth cond's assure it stays bdd in $W^{1,p}(\Omega)$ then there's a subsequence that converges wkly there. To show the wk limit is a minimizer one needs lower-semicontinuity, i.e.

$$u^j \rightharpoonup u^\infty \quad (\text{wkly in } W^{1,p}) \implies \int_\Omega W(Du^\infty) \le \liminf_{j\to\infty} \int_\Omega W(Du^j)$$

When W is convex the Fenchel transform provides a convenient proof: if

$$W(\xi) = \sup_\eta \langle \xi, \eta \rangle - W^*(\eta)$$

$$\implies \int_\Omega W(Du) = \sup_{\eta(x)} \int_\Omega \langle Du, \eta(x)\rangle - W^*(\eta(x))$$

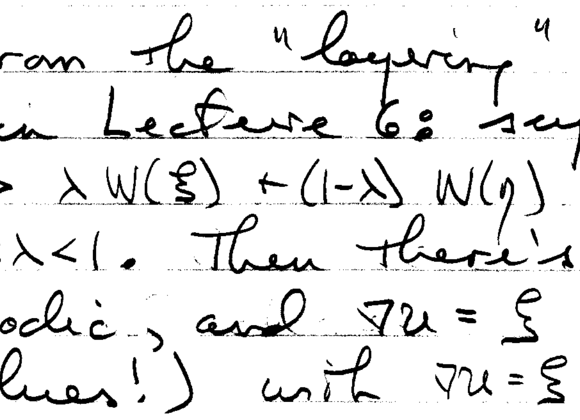
and RHS is sup of linear functionals. Since each linear fnl is cont's under wk convergence, the sup is lower semicont's.

Hypothesis of convexity is natural when u is <u>scalar</u>-valued, since

$$\int_\Omega W(Du)\,dx \quad \text{is lsc under wk}^* \ W^{1,\infty} \text{ convergence}$$

$\Rightarrow$ W must be convex.

This follows from the "layering" construction we discussed in Lecture 6: suppose $W(\lambda\xi + (1-\lambda)\eta) > \lambda W(\xi) + (1-\lambda)W(\eta)$ for some ξ, η in $\mathbb{R}^n$ and $0<\lambda<1$. Then there's a fn $u : \mathbb{R}^n \to \mathbb{R}$ st ∇u is periodic, and $\nabla u = \xi$ or η (just these two values!) with $\nabla u = \xi$ on fraction λ + $\nabla u = \eta$ on fraction $(1-\lambda)$.



(Note: $\nabla_t u$ must match (since u is cont's) for t parallel to layers, so layer normal $\vec{n} \parallel \xi - \eta$.)
Scaling this by $u^\varepsilon(x) = \varepsilon u(x/\varepsilon)$, we get that Du^ε converges wkly in L^∞ to $\lambda\xi + (1-\lambda)\eta$. So Lower semicontinuity $\Rightarrow$ $W(\lambda\xi + (1-\lambda)\eta)$
$\leq \lambda W(\xi) + (1-\lambda)W(\eta)$.

For vector-valued problems ($u: \mathbb{R}^m \to \mathbb{R}^n$, $m > 1$ and $n > 1$) things are different. In fact

a) now $\xi + \eta$ are matrices. Construction sketched above is possible only if $\xi - \eta$ has rank one, since

$\nabla_t u$ consistent across layers

$\Rightarrow \xi \cdot t = \eta \cdot t$ for t tangent to layers.

$\Rightarrow \xi - \eta = \vec{a} \otimes \vec{n}$ where $\vec{n}$ is layer normal

Thus: The argument sketched in pg 8.2 shows in general that

$$\text{lsc} \Rightarrow W(\lambda\xi + (1-\lambda)\eta) \le \lambda W(\xi) + (1-\lambda) W(\eta)$$

if $\xi - \eta$ has rank one

("W is rank-one convex").

b) there are, in fact, lsc functionals that are nonconvex. A simple example for maps $\mathbb{R}^2 \to \mathbb{R}^2$ is $\det(\nabla u)$. (To be explained soon)

c) some very natural examples require nonconvex W, eg nonlinear elasticity (to be explained soon).

Let's explain (b), focusing for simplicity on the case $u: \mathbb{R}^2 \to \mathbb{R}^2$.

The heart of the matter is that <u>det Du is wkly cont's (for $u: \mathbb{R}^2 \to \mathbb{R}^2$) under wk convergence in $W^{1,p}$ for $p>2$.</u>

More <u>careful statement</u>: note that since Du is 2×2, det Du is <u>quadratic</u> in Du. So if a sequence u^j stays bounded in $W^{1,p}(\Omega)$, then det(Du^j) is bounded in $L^{p/2}(\Omega)$. If $p>2$ then the seq is precompact in wk topology on $L^{p/2}$. So (for a subsequence) det Du^j converges wkly to <u>some</u> $L^{p/2}$ function. Our assertion is that the wk limit is det(Du^∞), if u^∞ is the weak limit.

Why is this useful? Because it gives a big supply of lsc functionals; in fact, if

$$W(Du) = \Phi(Du, \det Du)$$

with Φ a <u>convex</u> function of 5 vars, then

$\int_\Omega W(Du)\,dx$ is lsc under wk convergence in $W^p(\Omega)$ for $p > 2$.

Pf: use Fenchel transform of φ: if

$$\varphi(\xi, t) = \sup_{\eta, s} \eta \cdot \xi + s \cdot t - \varphi^*(\eta, s)$$

Then

$$\int_\Omega \varphi(Du, \det Du) = \sup_{\substack{\eta(x),\\ s(x)}} \int_\Omega \eta \cdot Du + s \cdot \det Du - \varphi^*(\eta, s)\,dx$$

$$= \sup \text{ of wkly cont's fns.}$$

(When $W(Du) = \varphi(Du, \det Du)$ we say "W is polyconvex".)

It remains to explain the wk cont'y of $\det Du$. We'll deduce it from the fact that

(*) <u>$\det Du$ can be expressed as a divergence</u>

How could we have guessed that? Well,

$$\int_\Omega \det Du = \text{area of image } u(\Omega), \text{ if } \det Du > 0$$

So integral depends only on bdry data. How else could that happen, but by $\det Du$ being a divergence?!

[Note corollary: Euler Lagrange eqn for $\int_\Omega \det Du$ must be an identity. Therefore $\det Du$ is said to be a "null-Lagrangian"].

Pf of (*), by exterior calculus:

$$d(u_1 \wedge du_2) = du_1 \wedge du_2 = (\det Du)\, dx_1 \wedge dx_2$$

Same proof, in coordinates:

$$\det(Du) = \partial_1 u_1 \partial_2 u_2 - \partial_1 u_2 \partial_2 u_1$$
$$= \partial_1 (u_1 \partial_2 u_2) - \partial_2 (u_1 \partial_1 u_2)$$

To see why (*) is useful, note that if Du^j stays bdd in L^p ($p>2$) + $u^j \to u^\infty$, we already know $\det Du^j$ has a limit in $L^{p/2}$ (wkly). To identify the limit it's suff to show $\det Du^j \to \det Du^\infty$ as distributions. So let g be smooth with cpt spt. Then

$$\int_\Omega (\det Du)\, g = \int_\Omega -u_1 \partial_2 u_2 \partial_1 g + u_1 \partial_1 u_2 \partial_2 g$$

So if $u^j \to u^\infty$ wkly $W^{1,p}$ then

$$\int_\Omega u_1^j \partial_2 u_2^j \partial_1 g \to \int_\Omega u_1^\infty \partial_2 u_2^\infty \partial_1 g$$

since $u_1^j \to u_1^\infty$ strongly in every L^q (using that dim = 2)
$\partial_2 u_2^j \to \partial_2 u_2^\infty$ wkly L^p (by hypothesis)

and similarly for the other term.

Extension of preceding discn to other dims ($u: \mathbb{R}^n \to \mathbb{R}^m$) involves no new ideas; each det of a $k \times k$ minor of Du is a null-Lagrangian. (See eg Dacorogna's book for more on polyconvexity.)

Let's turn now to pt (c) from pg (8.3): <u>nonlinear elasticity</u> provides an example where convexity is unacceptable as a structural hypothesis, but polyconvex energies provide a rich class of models. (I'll be brief; some places to read more: P. Ciarlet's book Mathematical Elasticity vol 1: Three-dimensional elasticity; or J Ball's 1977 article in Arch Rational Mech Anal 63(4) 337–403.)

Ω = undeformed body (assumed stress-free), a subset of $\mathbb{R}^n$ ($n=2,3$)

$u: \Omega \to \mathbb{R}^n$ deformation, takes $x \in \Omega$ to $u(x)$ = deformed position of the material originally at x

$F = Du$ is an $n \times n$ matrix (the "deformation gradient")

elastic energy $E[u] = \int_\Omega W(Du)\, dx$ has property that its 1st varn = 0 $\iff$ elastic equilibrium (with no applied forces); moreover stable equilibrium $\iff$ local min of E

Structural condns on W:

(1) $W(F) = W(RF)$ for all $R \in SO(n)$, since rotating a body should do no work (this condn is called "frame indifference"). By polar decomp, W depends only on $(F^T F)^{1/2}$

(2) if the elastic material is isotropic (eg rubber) then $W(F) = W(FR)$ for

$R\in SO(n)$; in this case W is a symmetric fn of the eigenvalues of $(F^TF)^{1/2}$ ("the principal stretches")

3) $W(F)$ is minimized at $F\in SO(n)$ (only!).
$W(F)\to\infty$ as $|F|\to\infty$
$W(F)\to\infty$ as $\det F\to 0$
(W need only be defined for $\det F>0$).

Key point for us: W cannot be convex, since $SO(n)$ is not convex. More concretely:

$n=2 \Rightarrow$ W cannot be convex since $W(I)=W(-I)=$ min value of W, but $W=\infty$ at $0=\frac{1}{2}I+\frac{1}{2}(-I)$

$n=3 \Rightarrow$ similar argument using I and $\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$

__Claim:__ there are plenty of __polyconvex__ W's meeting these requirements. Focus on 2D for simplicity: one can take, for example,

$$(*)\qquad W(F)=A(v_1^\alpha+v_2^\alpha)+g(v_1v_2)$$

where $A>0$, $\alpha>2$, v_1+v_2 are the eigenvalues of $(F^TF)^{1/2}$, and $2At^{\alpha/2}+g(t)$ is minimized (for $t\geq 0$) at $t=1$, and $g(t)$ is convex.

(In fact $V_1^\alpha + V_2^\alpha \geq 2(V_1V_2)^{\alpha/2}$ with equality only for $V_1 = V_2$; so
$$W(F) \geq 2A(V_1V_2)^{\alpha/2} + g(V_1, V_2)$$
is minimized when $V_1 = V_2 = 1$. The condition $\alpha > 2$ is meant to assure a uniform bound in $W^{1,p}$ for $p = \alpha > 2$; I guess I also need g to be bounded below, besides the conds stated above.)

The energy density (*) is polyconvex, though proof is not trivial. Clearly (since g is convex and $V_1V_2 = \det F$) it suffices to show that

$$F \xrightarrow[\Phi]{} A(V_1^\alpha + V_2^\alpha)$$ is a convex function of the matrix F

This is obvious if we restrict attn to diagonal F's, but not at all obvious when we remember that the convexity reln
$$\Phi(\lambda F + (1-\lambda)G) \leq \lambda \Phi(F) + (1-\lambda)\Phi(G)$$
is asserted even for $F + G$ that are not simultaneously diagonal. It is a special case of a more general result:

Thm: if $\Phi(F) = \varphi(v_1, \dots, v_n)$ where F is $n \times n$, $\{v_j\}$ are the eigenvalues of $(F^T F)^{1/2}$, and φ is a symmetric function of its arguments.

Then Φ is convex iff φ is convex and nondecreasing in each v_j.

(For a proof see Thm 5.1 of Ball's 1977 paper.)

For 3D rubber elasticity, a common hypothesis is incompressibility, and a commonly-used model is the "neoHookean" law

$$W(F) = A(v_1^2 + v_2^2 + v_3^2 - 3) \qquad v_1 v_2 v_3 = 1.$$

which is minimized at $v_1 = v_2 = v_3 = 1$ (by the arithmetic mean / geometric mean inequality). However it doesn't quite fit our setup since $\int W(Du)\,dx$ controls only the $W^{1,2}$ norm (not $W^{1,p}$ for $p > 2$).

Before leaving elasticity as a topic, let's spend a moment deriving <u>linear elasticity</u> from the nonlinear var'l pbms considered so far. This means we <u>restrict</u> attn to deformations near the identity

$$u(x) = x + \varepsilon w(x) \qquad \varepsilon \text{ small}$$

$$\Rightarrow (Du^T Du)^{1/2} = \left[(I + \varepsilon Dw)^T (I + \varepsilon Dw)\right]^{1/2}$$

$$= I + \varepsilon \frac{Dw + Dw^T}{2} + \text{higher order terms}$$

So what matters is the linear strain $e(w) = \frac{Dw + Dw^T}{2}$.

As for the energy:

$$W(I + \varepsilon Dw) = W(I) + (\text{linear term vanishes}) + \frac{1}{2}\varepsilon^2 \sum \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} D_\alpha w_i D_\beta w_j + O(\varepsilon^3)$$

Taking $W(I) = 0$ and recalling that $W(I + \varepsilon Dw)$ should depend only on $\left[(I + \varepsilon Dw)^T (I + \varepsilon Dw)\right]^{1/2}$ we see that

linear elastic energy should be a nonneg quadratic form in $e(w)$.

For an isotropic material the quadratic form should be a symmetric fn of the eigenvalues of $e(w)$; the general form is

$$\langle Ae, e\rangle = 2\mu |e|^2 + \lambda (\mathrm{tr}\, e)^2$$

and A is called the "Hooke's law" of the material. Thus, for example, the linear elasticity prbm with fixed displacements at $\partial\Omega$ is

$$\min_{\substack{w = w_0 \\ \text{at } \partial\Omega}} \frac{1}{2}\int_\Omega \langle A\, e(w), e(w)\rangle \, dx$$

and the problem with specified traction (force per unit area) at $\partial\Omega$ is

$$\min \frac{1}{2}\int_\Omega \langle Ae(w), e(w)\rangle \, dx - \int_{\partial\Omega} \langle w, f\rangle \, dA$$

where $f : \partial\Omega \to \mathbb{R}^n$ is the traction at the bdry,

<u>Suggested exercises</u>:

(1) We showed that if $u^j \to u^\infty$ wkly in $W^{1,p}(\Omega)$, $p > 2$ (for maps $\mathbb{R}^2 \to \mathbb{R}^2$; by defn this means $\int_\Omega |Du^j|^p dx \le C$ with C indep of j, and $\int_\Omega \langle Du^j, \varphi\rangle \to \int_\Omega \langle Du^\infty, \varphi\rangle$ for any $L^{p'}$ vector field φ) then $\det Du^j \to \det Du^\infty$

wkly $L^{p/2}$. When $\{u^j\}$ are more singular strange things can happen due to "cavitation". Explore this by considering the map $u^j : B_1 \to B_1$ (where B_1 = unit ball of $\mathbb{R}^2$) such that

$$u^j(x) = \begin{cases} \dfrac{1-\rho_j}{\rho_j}\, x & \text{for } |x| \le \rho_j \\ \left(\dfrac{1-2\rho_j}{1-\rho_j} + \dfrac{\rho_j}{1-\rho_j}|x| \right) \dfrac{x}{|x|} & \text{for } \rho_j \le |x| \le 1 \end{cases}$$

so u^j "blows up" B_{ρ_j} to $B_{1-\rho_j}$ and "squashes" $B_1 \setminus B_{\rho_j}$ to $B_1 \setminus B_{1-\rho_j}$. (Here ρ_j can be any sequence converging to zero.)

a) Show that $u^j(x) \to \dfrac{x}{|x|}$ ae as $j \to \infty$.

b) Show that $\det Du^j$ converges as a distribution to a point mass located at 0

c) For which p does $\int_B |Du^j|^p dx$ stay bounded as $j \to \infty$?

[Comment: for additional examples, some similar and others rather different, see §7 of J Ball + F Murat, J Funct Anal 58, 1984, 225-253. That section is indep of the rest of the paper.]

(2) Show that if $\int_\Omega g(\det Du)\,dX$ is lower semicontinuous (for $\Omega \subset \mathbb{R}^2$ + $u : \Omega \to \mathbb{R}^2$) then g must be convex.

<u>Hint</u>: it suffices to show that
$g(\theta a + (1-\theta) b) \le \theta g(a) + (1-\theta) g(b)$ for $0<\theta<1$
and $a, b \in \mathbb{R}$. Do this by considering $u_\varepsilon(x,y) = (u_{1\varepsilon}, u_{2\varepsilon})$ defined by

$$u_{1\varepsilon}(x,y) = (\theta a + (1-\theta) b)\,x + \varepsilon \varphi(x/\varepsilon)$$

$$u_{2\varepsilon}(x,y) = y$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is periodic with period 1 such that

$$\varphi(t) = \begin{cases} t(1-\theta)(a-b) & 0<t<\theta \\ (1-t)\theta(a-b) & \theta<t<1 \end{cases}.$$

(3) It can be nontrivial to determine whether a given function $W(F)$ is polyconvex or not. As an example, show that in the 2×2 setting

$$W(F) = \begin{cases} 1 + |F|^2 & \text{if } \rho(F) \ge 1 \\ 2\rho(F) - 2|\det F| & \text{if } \rho(F) \le 1 \end{cases}$$

is polyconvex, when $|F|^2 = \sum_{i,j=1}^{2} |F_{ij}|^2$ and

$$\rho(F) = (|F|^2 + 2|\det F|)^{1/2}.$$

(This example arose as the "relaxation" of a nonconvex problem from "optimal design", in my work with Strang in the 80's.)

<u>Hint</u>: show that $W(F) = g(F, \det F)$ where

$$g(F,t) = \max_{\alpha = \pm 1} \left\{ f\left(\left[|F|^2 + 2\alpha \det F \right]^{1/2} \right) - 2\alpha t \right\}.$$

in which

$$f(t) = \begin{cases} 1+t^2 & t \geq 1 \\ 2t & t \leq 1 \end{cases},$$

Then check that $g(F,t)$ is a convex function of F and t.