Calculus of Variations, Lecture 5, 2/27/2017

New topic: 1D problems

$$\min \int_a^b F(t, u(t), \dot{u}(t))\, dt \qquad u : [a,b] \to \mathbb{R}^n$$

emphasizing

a) geodesics, as a key example
b) importance of F being convex wrt $\dot{u}$
c) role of 2^{nd} variation; conjugate pts

[Students taking Mechanics will see additional examples there, associated with solving eqns of Hamiltonian mechanics by "action minimization".]

Reasonable source for most of this material: Jost + Li-Jost, Sections 1.1-1.3 and 2.1.

Key example: geodesics. By defn: a geodesic is a curve that (locally) minimizes arc length. In local coordinates, if the curve is $\vec{x}(t)$,

$$|dx| = |\dot{x}(t)|\, dt = \left(\sum g_{ij}(x(t))\, \dot{x}_i \dot{x}_j\right)^{1/2}$$

where g_{ij} is the Riemannian metric; assoc var'l problem is

$$L = \int_a^b |\dot{x}(t)|\, dt$$

(note: we're interested in <u>critical</u> <u>pts</u>, not just minima).

Two issues:

(1) this has the form $\int F(x(t), \dot{x}(t))\,dt$ but F is not smooth in $\dot{x}$ near $\dot{x}=0$

(2) arc length is indep of parametrization, so var'l pbm chooses a "curve" but not any particular parametrization (thus: a dramatic but geometrically-natural failure of uniqueness)

Both issues can be fixed by considering instead the <u>different</u> functional

$$E = \frac{1}{2}\int_a^b |\dot{x}|^2\,dt$$

where $|\dot{x}|^2 = \sum g_{ij}(x(t))\,\dot{x}_i(t)\,\dot{x}_j(t)$. To see why, observe that for any parametrized curve $\vec{x}(t)$,

$$L[x] \leq \sqrt{2(b-a)}\,\sqrt{E[x]}$$

(with strict inequality unless $|\dot{x}|$ is constant), as a consequence of

$$\int_a^b |\dot{x}|\,dt \leq \left(\int_a^b |\dot{x}|^2\,dt\right)^{1/2}\left(\int_a^b 1\,dt\right)^{1/2}$$

Thus

$$\text{min value of } L \leq \sqrt{2(b-a)} \cdot (\text{min value of } E)^{1/2}.$$

But opposite $\neq$ is easy: given any curve with length ℓ, its constant-speed parametrization has $|\dot{x}| = \ell/(b-a)$, so

$$\frac{1}{2}\int_a^b |\dot{x}|^2\,dt = \frac{1}{2}(b-a)\frac{\ell^2}{(b-a)^2} = \frac{1}{2(b-a)}\ell^2$$

Thus

$$(\text{min value of } E)^{1/2} \leq \frac{1}{\sqrt{2(b-a)}}(\text{min value of } L)$$

<u>Conclusion</u>: minimizer of E has min length <u>and</u> constant speed.

(Exercise: use the EL eqn for E to give a different proof that extremals – even critical pts! – of E have constant speed, by showing that $\frac{d}{dt}|\dot{x}|^2 = 0$ if $x(t)$ solves the EL eqn.).

<u>Key properties of geodesics</u>:

a) they're smooth
b) they're locally paths of shortest length
c) globally, they may <u>not</u> be paths of shortest length (eg on a sphere the geodesics are arcs of great circles)

<u>Rmk</u>: discn above assumed we had a single "coordinate chart" valid along entire curve. Locally true, but not necessarily globally so. In general must use different coord charts on different parts of curve (see §2.1 of Jost/Li-Jost for detail on what this means).

Properties (a) – (c) are <u>not</u> special to geodesics; so it's natural to discuss them more generally, for pblms of form

$$\int_a^b F(t, u(t), \dot{u}(t))\,dt$$

where $u : [a,b] \to \mathbb{R}^n$. Note that EL eqn in this setting is

$$\frac{\partial F}{\partial u_j} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}_j} = 0 \qquad 1 \le j \le n.$$

<u>Discussion of (a) = smoothness of solns</u>: we clearly need some condition on F, since for $u : [-1,1] \to \mathbb{R}$,

$$\min_{\substack{u(-1)=0\\ u(1)=0}} \int_{-1}^{+1} (u_t^2 - 1)^2\,dt$$ is solved by 

$$\min_{\substack{u(-1)=0\\ u(1)=1}} \int_{-1}^{+1} (u_t - 1)^2 u^2\,dt$$ is solved by 

Convenient hypothesis is that $F(t,u,p)$ is smooth enough (I won't try to give minimal conds - see Jost + Li-Jost for such things) and strictly convex in p. The point: the EL eqn can be written as

$$\frac{\partial F}{\partial u_j} - \frac{\partial^2 F}{\partial \dot{u}_j \partial t} - \sum_k \frac{\partial^2 F}{\partial \dot{u}_j \partial u_k} \dot{u}_k = \sum_k \frac{\partial^2 F}{\partial \dot{u}_j \partial \dot{u}_k} \ddot{u}_k ,$$

which we can solve (inverting the strictly pos. def. matrix $\partial^2 F / \partial \dot{u}_j \partial \dot{u}_k$) to see that $\ddot{u}$ is bounded if $\dot{u}$ is bounded. Higher derivs can be handled similarly (differentiate eqn in t).

Preceding arg't is a bit sloppy, since it assumes $\ddot{u}$ exists. Let's explain why strict convexity $\Rightarrow$ it must exist. Consider

$$\varphi_j(t,u,p,q) = \frac{\partial F}{\partial p_j} - q_j$$

and observe that $\vec{p}$ solves $\vec{\varphi}(t,u,p,q) = 0$ iff it achieves

$$\max_p \langle q, p \rangle - F(t,u,p) .$$

(Here t, u, q enter only as parameters; maximizing p is unique if F is strictly convex; arg't has an implicit hypothesis that $F(t,u,p)$ grows faster than linearly as $|p| \to \infty$, so optimal p in preceding formula exists [$p \to \infty$ is not optimal].)

Implicit fn Thm + hypothesis that $\frac{\partial^2 F}{\partial p_j \partial p_k}$ has full rank $\Rightarrow$ we can (locally) solve eqn $\vec{\Phi} = 0$ for p as fn of other vars, say

$$\frac{\partial F}{\partial p_j} = q_j \quad \forall j \quad \Longleftrightarrow \quad p_j = \psi_j(t, u, \vec{q}).$$

Now, we know $\vec{\Phi} = 0$ when $\vec{q} = \frac{\partial F}{\partial \vec{p}}$. Evaluating this at $p = \dot{u}$ gives

$$\dot{u}_j(t) = \psi_j\left(t, u(t), \frac{\partial F}{\partial \dot{u}}(t, u, \dot{u})\right)$$

RHS is diff'ble (using EL eqn to know differentiability of $\partial F / \partial \dot{u}$) so LHS is diff'ble (in t).

Rest of these notes discusses pts (b) + (c) (local minimality, conjugate pts, etc).

<u>Brief summary</u>:

1) 2^{nd} variation provides a convenient necessary condition for minimality

2) importance of convexity is visible here too: if F is not convex in p, then 2^{nd} varn test is sure to fail.

3) as we work on longer time intervals (eg $[a,b]$, with a fixed + $b\uparrow$) failure of local optimality can be detected by 2nd varn test

(Sufft conds for optimality are also interesting of course, but they would lead us too far astray.)

Defn of 2nd varn: given a soln $u(t)$ of the EL eqn, it is natural to consider

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_a^b F(t, u+s\eta, \dot{u}+s\dot{\eta})\, dt$$

where $\vec{\eta}(t)$ is arbitrary (except perhaps for restrs assoc to bdry conds). This reduces to

$$Q[\eta] = \int_a^b F_{uu}\,\eta\otimes\eta + 2F_{up}\,\eta\otimes\dot{\eta} + F_{pp}\,\dot{\eta}\otimes\dot{\eta}\; dt$$

where, for example,

$$F_{uu}\,\eta\otimes\eta = \sum \frac{\partial^2 F}{\partial u_i \partial u_j}(t,u,\dot{u})\, \eta_i(t)\eta_j(t).$$

Focusing on case when $u(a)$ + $u(b)$ are fixed (so $\eta(a)=\eta(b)=0$) we see that

$$u \text{ is loc min} \implies Q[\eta] \geq 0 \text{ for all } \eta \text{ st } \eta(a)=\eta(b)=0.$$

Importance of convexity is 2-fold.

First: If $F_{pp} \geq c_0 I$ with $c_0 > 0$ (this is a little stronger than strict convexity) then

$$F_{pp}\,\dot\eta\otimes\dot\eta \geq c_0 |\dot\eta|^2$$

and it's easy to see that Q is strictly positive if $|b-a|$ is small enough. (Hint: $\int_a^b |\eta|^2 \leq C(b-a)^2 \int_a^b |\dot\eta|^2$ if $\eta(a) = \eta(b) = 0$, with C indep of b, a.)

Second: If, for some t_0, the matrix

$$\frac{\partial^2 F}{\partial p_i \partial p_j}(t_0, u(t_0), \dot u(t_0))$$

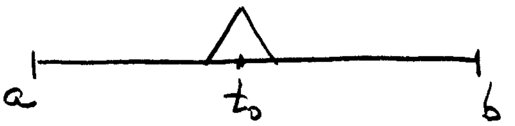
is not ≥ 0 (roughly: F is not convex in p at some point along the curve) then $\exists\, \eta$ st $Q[\eta] < 0$. In fact, it suffices to choose $\xi \in \mathbb{R}^n$ st

$$\sum \frac{\partial^2 F}{\partial p_i \partial p_j}(t_0, u(t_0), \dot u(t_0))\, \xi_i \xi_j < 0$$

and then take $\eta(t)$ supported in a (small) nbhd of t_0 st

$$\dot\eta(t) \in \{0, \pm\xi\}$$

Picture:



$$\dot\eta = \begin{cases} \xi & t_0 - \varepsilon < t < t_0 \\ -\xi & t_0 < t < t_0 + \varepsilon \end{cases}$$

If ε is small enough then $Q[\eta]$ is strictly negative (since $\int F_{pp}\,\dot\eta\otimes\dot\eta\,dt$ scales like ε and is negative, while the other terms in Q scale like ε^2).

<u>On long "time intervals" local optimality can be lost.</u>

Recall example of geodesics on a sphere!

We can detect loss of optimality using the 2nd varn quadratic form.

Observe that it makes sense to ask: <u>does EL</u> <u>for</u>

$$\min_{\substack{\eta(a)=0\\ \eta(b)=0}} \int_a^b F_{uu}\,\eta\otimes\eta + 2F_{up}\,\eta\otimes\dot\eta + F_{pp}\,\dot\eta\otimes\dot\eta \; dt$$

<u>have a nonzero soln $\eta(t)$</u>? (Such $\eta(t)$ solves the "homogeneous" 2nd order ODE

$$F_{uu}\eta + F_{up}\dot\eta - \frac{d}{dt}(F_{up}\eta) - \frac{d}{dt}(F_{pp}\dot\eta) = 0$$

and is called a "Jacobi field".) If such $\eta(t)$ exists, we say <u>b is conjugate to a</u>.

<u>Thm</u>: After 1st conjugate pt, an extremal (ie a soln of EL eqn) ceases to be a minimizer.

Pf: Let b_0 be conjugate to a, and $b > b_0$.
Try

$$(*) \quad \eta = \begin{cases} \text{nonzero Jacobi field on } (a, b_0) \\ 0 \qquad\qquad\qquad\qquad \text{on } (b_0, b) \end{cases}$$

I claim that <u>2nd var'n evaluated at this η vanishes.</u>

Accepting this for a moment, the rest is easy: the η just defined is not C^2, so it cannot minimize the 2nd var'n fn'l (note: we have assumed that $F_{pp} > 0$ so the term $F_{pp}\,\dot\eta \otimes \dot\eta$ in the 2nd var'n quadr form is strictly convex). Thus $\exists$ some <u>other</u> $\tilde\eta(t)$ for which 2nd varn is negative.

To see why 2nd varn eval at $(*)$ vanishes, consider

$$\varphi(t, \eta, \dot\eta) = F_{uu}\,\eta \otimes \eta + 2F_{up}\,\eta \otimes \dot\eta + F_{pp}\,\dot\eta \otimes \dot\eta$$

and observe that

$$\varphi(t, \alpha\eta, \alpha\dot\eta) = \alpha^2 \varphi(t, \eta, \dot\eta).$$

So (by diff'n wrt α at $\alpha = 1$)

$$\eta \cdot \varphi_\eta + \dot\eta \cdot \varphi_{\dot\eta} = 2\varphi$$

Integrating:

$$\int_a^b \varphi(t,\eta,\dot\eta) = \int_a^{b_0} \varphi(t,\eta,\dot\eta)\ dt$$
$$= \frac{1}{2}\int_a^{b_0} \eta\varphi_\eta + \dot\eta\varphi_{\dot\eta}\quad dt$$
$$= \frac{1}{2}\int_a^{b_0} \eta\left(\varphi_\eta - \frac{d}{dt}\varphi_{\dot\eta}\right) dt$$

using that $\eta(a) = \eta(b_0) = 0$. But restrn of η to $[a, b_0]$ minimizes the 2nd varn quadr form on $[a, b_0]$ (giving min value 0), so it solves the EL eqn

$$\varphi_\eta - \frac{d}{dt}(\varphi_{\dot\eta}) = 0 \qquad \text{on } [a, b_0].$$

Therefore

$$\int_a^b \varphi(t,\eta,\dot\eta)\ dt = 0$$

as asserted.

I only proved in these notes that 2nd varn > 0 on short intervals. But it's also true that crit pts are minimizers on short intervals if F_{pp} is pos def.

For special case of geodesics on $S^2 \subset \mathbb{R}^3$, the antipodal pt is the conjugate pt. To see why, observe that

a) for antipodal pts, there is a 1-par family of shortest paths (great semicircles)

b) if there's a 1-par family of minimizers $u^\theta(t)$ then $\eta = \frac{\partial}{\partial\theta} u^\theta$ is a Jacobi field.

Point (a) is obvious. For (b): each u^θ solves EL eqn

$$F_u(t, u^\theta, \dot{u}^\theta) = \frac{d}{dt} F_p(t, u^\theta, \dot{u}^\theta).$$

Diffn wrt to θ $\Rightarrow$

$$F_{uu}\,\eta + F_{up}\,\dot{\eta} = \frac{d}{dt}\left(F_{pu}\,\eta + F_{pp}\,\dot{\eta}\right)$$

with $\eta = \frac{\partial}{\partial\theta} u^\theta$. This is precisely the eqn characterizing a Jacobi field (note that $\eta = 0$ at endpoints, ie poles).

[More conceptual pf: value of $\int_a^b F(t, u^\theta, \dot{u}^\theta)\,dt$ indep of θ $\Rightarrow$ both 1st and 2nd derivs vanish wrto θ. So (assuming u^θ is extremal) $\eta = \frac{\partial}{\partial\theta} u^\theta$ achieves value 0 in the 2nd vari'n fnl. Since the min value is zero, this η must be a Jacobi field.]

Suggested Exercises:

(1) Show directly (using the EL eqns) that any extremal for $\int_a^b |\dot{x}|^2 dt$ has constant speed (see pp 2-3).

(2) Show that if b is conjugate to a then

$$\min_{\substack{\eta(a)=0\\ \eta(b)=0}} \int_a^b F_{uu}\,\eta\otimes\eta + 2F_{up}\,\eta\otimes\dot{\eta} + F_{pp}\,\dot{\eta}\otimes\dot{\eta} = 0$$

(3) When studying waves it is useful to consider paths that minimize "travel time," where the wave speed $\gamma(x)$ is a known function of location x. Show that this amounts to considering geodesics in the metric

$$g_{ij} = \frac{1}{\gamma(x)}\,\delta_{ij} .$$

(4) In these notes we focused on Dir bc, ie $\min \int_a^b F(t,u,\dot{u})\,dt$ subject to $u(a)=\alpha$ and $u(b)=\beta$ being given. Suppose instead we impose $u(a)=\alpha$ and $u(b)\in M$ where M is a submanifold. What end condition does the EL get at $t=b$? What is the proper notion of Jacobi field in this case?

(5) Show that the only critical pts of

$$\int_a^b u_x^2 + (u^2-1)^2$$

(no boundary condition!) with nonnegative 2nd variation are the "trivial ones," namely $u \equiv -1$ and $u \equiv +1$. (Hint: let u be a critical point. Show that $\eta = u_x$ achieves value 0 in the 2nd variation quadratic form. Then argue that this η can't be a minimizer of that quadratic form.)